Calculus 1
Mathematiques

2023/2024

Chapiter 1

The Real Line

Exercise 1

Show that, for all $(x,y,u,v)\in\mathbb{R}^4$

1. $|x+y|\leq|x|+|y|$.
2. $|x-y|\geq||x|-|y||$.
3. $||x|-|y||\leq|x|+|y|$ then deduce that : $||x-u|-|y-v||\leq|x-u|+|y-v|$
4. $|ux-vy|\leq\sqrt{(x^2+y^2)(u^2+v^2)}$.

Exercise 2

Let $A,\ B$ be two non empty and bounded subsets of $\mathbb{R}$. Proove that :

1. If $A\subset B$ then $\inf B\leq\inf A$ and $\sup A\leq\sup B$.
2. $\sup(A\cup B)=\max(\sup A,\ \sup B)$.
3. $\inf(A\cup B)=\min(\inf A,\ \inf B)$.
4. If $A\cap B\neq\varnothing$ then :
   (a) $\sup(A\cap B)\leq\min(\sup A,\sup B)$.
   (b) $\inf(A\cap B)\geq\max(\inf A,\inf B)$.
5. Put : $A+B=\{x+y:x\in A$ and $y\in B\}$, Show that :
   (a) $\sup(A+B)=\sup A+\sup B$.
   (b) $\inf(A+B)=\inf A+\inf B$.

Exercise 3

Let be the following set : $E=\left\{\dfrac{\cos(m\pi)}{m}+e^{-|n|}:\ n\in\mathbb{Z}\text{ and }m\in\mathbb{Z}^*\right\}$

1. Show that E is bounded .
2. Find, if they exist, the quantities : $\sup E,\ \inf E,\ \max E,\ \min E$.

Exercise 4

Let be the following set : $E=\left\{\dfrac{1}{(-1)^m m+(-1)^n n}:\ (n,m)\in\mathbb{N}^*\times\mathbb{N}^*\right\}$.

1. Show that $\forall(n,m)\in\mathbb{N}^*\times\mathbb{N}^*:(-1)^m m+(-1)^n n\neq0$.

2. Find, if they exist : $\sup E, \inf E, \max E, \min E$.

Exercise 5

Find the supremum, the maximum, the infimum and the minimum of the following sets, whenever they exist :

1. $E_1 = \left\{ \frac{2}{n+1} : n \in \mathbb{N} \right\}$.
2. $E_2 = \left\{ 1 + \frac{n}{3} : n \in \mathbb{N} \right\}$.
3. $E = \left\{ \frac{n^2-1}{n^2+1} : n \in \mathbb{N} \right\}$ (posed as an exam problem)
4. $E_3 = \left\{ (-1)^n + \frac{1}{n} : n \in \mathbb{N}^* \right\}$.
5. $E_4 = \left\{ \sqrt{n+1} - \sqrt{n} : n \in \mathbb{N} \right\}$.
6. $E_5 = \left\{ \frac{1}{n} + \sin\left(\frac{2n\pi}{3}\right) : n \in \overset{*}{\mathbb{N}} \right\}$.
7. $E_6 = \left\{ \cos\left(\frac{2n\pi}{3}\right) + \sin\left(\frac{2m\pi}{5}\right) : (n,m) \in \mathbb{Z}^2 \right\}$. (Additional)

Exercise 6

1. Show that :
   (a) $\forall n \in \mathbb{Z}, \forall x \in \mathbb{R} : E(n+x) = n + E(x)$.
   (b) $\forall (x,y) \in \mathbb{R}^2 : E(x+y) - E(x) - E(y) \in \{0,1\}$.
   (c) $\forall n \in \overset{*}{\mathbb{N}}, \forall a \in \mathbb{R} : a < \frac{E(na)+1}{n} \leq a + \frac{1}{n}$.
   (d) $\forall n \in \mathbb{N}^*, \forall x \in \mathbb{R} : E\left(\frac{E(nx)}{n}\right) = E(x)$.

Exercice 7 (additional)

Let be the following set :

$$E = \left\{ \frac{n+1}{n+2} \sin\left(\frac{1}{x}\right) : n \in \mathbb{N} \text{ and } x \in [-1,1] - \{0\} \right\}$$

1. Put :

$$A = \left\{ \frac{n+1}{n+2} : n \in \mathbb{N} \right\} \text{ and } B = \left\{ \sin\left(\frac{1}{x}\right) : x \in [-1,1] - \{0\} \right\}$$

   (a) Chek that A and B are bounded .
   (b) Find : $SupA, \inf A, SupB$ and $\inf B$.
2. Show that E is bounded, then find : $SupE$ and $\inf E$.
3. Do $MaxE$ and $\min E$ exist ?