Chapiter 1	The real line

1st Grade
1st Semester
Calculus 1
Mathematics
Ms RAHMOUN

1/ Vocabulary

مستقيم الأعداد الحقيقية	The real line	la droite réelle
الأعداد الحقيقية	Real numbers	Nombres réels
الأعداد الطبيعية	Naturel numbers	les entiers naturels
الأعداد الصحيحة	Integer numbers	les entiers relatifs
الأعداد الكسرية	Rational numbers	les rationnels
القاسم المشترك الأكبر	Greatest Common Divisor	Plus Grand Commun Diviseur
الأعداد اللاناطقة	Irrational numbers	les irrationnels
علاقة ترتيب	Order relation	Relation d'ordre
الإشارة	Sign	Signe
غير خالي	non empty	non vide
مجموعة جزئية	Subset	sous-ensemble
حدّ أعلى - حدّ أدنى	Upper bound - lower bound	majorant - minorant

2/ Recalls

∘/ $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of naturel numbers.

∘/ $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of integers.

∘/ $\mathbb{Q} = \{ \frac{m}{n} \,/\, m \in \mathbb{Z} \text{ and } n \in \mathbb{Z}^* \}$ is the set of rationals.

⚠ Note that :-) any rational has a "representative" $\frac{a}{b}$ such that $a \in \mathbb{Z}$, $b \in \mathbb{Z}^*$ and $GCD(a, b) = 1$

:-) A rational number has got either a finet number of decimals or an infinet number of decimals that are periodic.

Examples

∘/ $\frac{1}{3} = 0{,}3333\dots$ infinet decimals but periodic.

So $\frac{1}{3} \in \mathbb{Q}$

∘/ $\frac{5}{7} = 1{,}4$ finet decimals, so $\frac{5}{7} \in \mathbb{Q}$

∘/ $\mathbb{R} = \mathbb{Q} \cup \{\text{irrational}\}$ is the set of real numbers.

⚠ irrational numbers are those where decimals are infinet and non_periodic as $\sqrt{2}$, π, e, ...

•/ $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

3/ The real line is ordered by $\leq$

Let us consider the set of real numbers $\mathbb{R}$, provided with the (naturel) addition and multiplication (of numbers) and the relation $\leq$: $((\mathbb{R}, +, \cdot), \leq)$

a) $\leq$ is an order relation

That is to say that $\leq$ is :

1/ Reflexive i.e $\forall x \in \mathbb{R}$, $x \leq x$

2/ Anti-Symmetric i.e $\forall x, y \in \mathbb{R}$,

$$\begin{cases} x \leq y \\ y \leq x \end{cases} \Rightarrow x = y$$

3/ Transitive i.e $\forall x, y, z \in \mathbb{R}$

$$\begin{cases} x \leq y \\ y \leq z \end{cases} \Rightarrow x \leq z$$

b) $\leq$ is a total order relation on $\mathbb{R}$

Which means that $\forall x, y \in \mathbb{R}$ either $x \leq y$ or $y \leq x$

we say that $\leq$ is totaly ordering $\mathbb{R}$ or that $\mathbb{R}$ is totaly ordered by $\leq$.

c) Propreties

1. $\forall x, y, z \in \mathbb{R}$ · $x \leq y \iff x + z \leq y + z$

2. $\forall x, y \in \mathbb{R}$, $\begin{cases} \forall z \in \mathbb{R}^{+}_{*}, & x \leq y \iff x \cdot z \leq y \cdot z \\ \forall z \in \mathbb{R}^{-}_{*}, & x \leq y \iff x z \geq y \cdot z \end{cases}$

3. $\forall x, y \in \mathbb{R}^{*}$ / $\underbrace{x \cdot y > 0}_{\text{i.e } x \text{ and } y \text{ are of the same sign}}$; $x \leq y \iff \frac{1}{x} \geq \frac{1}{y}$

Note i.e. is the acronym of id est, a latten word that means « that is to say »

4. $\forall x \in \mathbb{R}, \begin{cases} \forall y \in \mathbb{R}_*^+, & x \leq y \Rightarrow \frac{x}{y} \leq 1 \\ \forall y \in \mathbb{R}_*^-, & x \leq y \Rightarrow \frac{x}{y} \geq 1 \end{cases}$

5. $\forall x, y, x', y' \in \mathbb{R},$

$$\begin{cases} x \leq y \\ x' \leq y' \end{cases} \Rightarrow x + x' \leq y + y'$$

d) Beware of these mistakes. (Common among students)

1. $\begin{cases} x \leq y \\ x' \leq y' \end{cases} \Rightarrow \cancel{x - x' \leq y - y'}$ Example $\begin{cases} -2 \leq 1 \\ -6 \leq -2 \end{cases}$
But $\underbrace{(-2)-(-6)}_{=4} \leq \underbrace{1-(-2)}_{=3}$

2. $\begin{cases} x \leq y \\ x' \leq y' \end{cases} \Rightarrow \cancel{x \cdot x' \leq y \cdot y'}$ Example.
$-2 \leq 1$
$-4 \leq -3$
But $\underbrace{(-2)(-4)}_{=8} \cancel{\leq} \underbrace{1(-3)}_{=-3}$ $>$

3. $x \leq y \Rightarrow \cancel{\frac{1}{x} \geq \frac{1}{y}}$

x and y should be of the same sign.

Example $-2 \leq 1$ but $-\frac{1}{2} \leq 1$ still.

4/ The absolute value

a) Definition: Let $x \in \mathbb{R}$. We define the absolute value of x by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

b) Properties:

1. $\forall x \in \mathbb{R}, \quad |x| = 0 \Leftrightarrow x = 0$
2. $\forall x, y \in \mathbb{R}, \quad |x \cdot y| = |x| \cdot |y|$
3. $\forall x, y \in \mathbb{R}, \quad |x + y| \leq |x| + |y|$ ← First triangular inequality.
4. $\forall x, y \in \mathbb{R}, \quad |x - y| \geq \big| |x| - |y| \big|$ ← Second triangular inequality.
5. $\forall x \in \mathbb{R}, \quad \sqrt{x^2} = |x|$
6. $\forall x \in \mathbb{R}, \begin{cases} \forall a \in \mathbb{R}^+, & |x| \leq a \Rightarrow -a \leq x \leq a \\ \forall a \in \mathbb{R}^-, & |x| \leq a \text{ is impossible.} \end{cases}$ $S = [-a, a]$

$S = \emptyset$

7. $\forall x \in \mathbb{R}$, $\begin{cases} \forall a \in \mathbb{R}^+ & |x| \geq a \Leftrightarrow x \geq a \text{ or } x \leq -a \\ \forall a \in \mathbb{R}^- & |x| \geq a \text{ is obvious} \end{cases}$

$S =]-\infty, -a] \cup [a, +\infty[$.

$-a \quad 0 \quad a$

$S = \mathbb{R}$

Exercises

AT home

1. Show that $\forall a, b, c \in \mathbb{R}$
   - $ab \leq \frac{a^2+b^2}{2}$
   - $ab + bc + ac \leq a^2 + b^2 + c^2$
2. Solve in $\mathbb{R}$
   - $|x+3| \leq 5$
   - $|x+2| > 7$
3. Solve $\left|\frac{1}{x} - 2\right| \leq 3$ in $\mathbb{R}^*$.

At the amphiteater

1. Let $x \in [3, 6]$ and $y \in [-4, -2]$ frame the Quantities $x+y$, $x-y$, xy, $\frac{x}{y}$.
2. Solve in $\mathbb{R}$
   $|2x-4| \leq |x+2|$

   Show that:
3. $|x| + |y| \leq |x+y| + |x-y|$
   $\forall x, y \in \mathbb{R}$

5/ Upper bounds - lower bounds

a) Definitions Let E be an non empty subset of $\mathbb{R}$

1. An element M of $\mathbb{R}$ is said to be an upper bound of E if and only if (iff) $\Leftrightarrow$ $\forall x \in E, \ x \leq M$.
2. An element m of $\mathbb{R}$ is said to be a Lower bound of E iff $\forall x \in E, \ m \leq x$
3. E is said to be bounded from above (or upper bounded) iff $\Leftrightarrow$ it admits an upper bound. (at least one)
4. E is said to be bounded from below (or Lower bounded) iff $\Leftrightarrow$ it admits a lower bound. (at least one).
5. E is said to be bounded iff it is upper bounded and lower bounded at the same time, i.e. $\exists \alpha, \beta \in \mathbb{R} / \forall x \in E, \alpha \leq x \leq \beta$.

b) Examples:

- $E_1 = \{1, 2, 3\}$ 3 is an upper bound of E_1 because $\forall x \in E_1, \ x \leq 3$, but so is 5, 6, ...

So, the set of all possible upper bounds of E_1, denoted $M(E_1)$ is: $M(E_1) = [3, +\infty[$

1 is a lower bound of E_1, because $\forall x \in E_1$, $x \geq 1$.
but so is 0 and (-1) ...

So, the set of all possible lower bounds of E_1 is
$m(E_1) =]-\infty, 1]$ ⚠ E_1 is bounded cause $\forall x \in E_1$ $1 \leq x \leq 3$

- $E_2 = [0, 2[$ $\quad M(E_2) = ?$ $\quad m(E_2) = ?$

well, $M(E_2) = [2, +\infty[$ and $m(E_2) =]-\infty, 0]$. ⚠ E_2 is bounded

- $E_3 =]0, +\infty[$.

E_3 does not admit an upper bound. So $M(E_3) = \phi$
$m(E_3) =]-\infty, 0]$. E_3 is not bounded, it is Only bounded from below.

- $E_4 = \{ \cos x, x \in \mathbb{R} \}$. $M(E_4) = [1, +\infty[$; $m(E_4) =]-\infty, -1]$

⚠ E_4 is bounded.

Note To show that a set E is bounded, we often use the following definition $\exists \gamma \in \mathbb{R}^+ / \forall x \in E \quad |x| \leq \gamma$

c) Max and Min

E is a non empty subset of $\mathbb{R}$.

1. M is the maximum of E (denoted $\max E$) iff:

M is an upper bound of E and $M \in E$

We say that $\max E$ is the greate element of E. When it exists, it is unique.

2. m is the minimum of E (denote $\min E$) iff:

m is a lower bound of E and $m \in E$

We say that $\min E$ is the lowest element of E. When it exists, it is unique.

Examples Let's get back to our examples $E_1, \dots, E_4$, and try to find $\max E_i$, and $\min E_i$ each time.

- $\min E_1 = 1$, $\max E_1 = 3$
- $\min E_2 = 0$, $\max E_2$ doesn't exist.
- $\min E_3$ and $\max E_3$ don't exist.
- $\max E_4 = 1$ because $1 = \cos 0$, $0 \in \mathbb{R}$, So, $1 \in E_4$

$\min E_4 = -1$ because $-1 = \cos \pi$, $\pi \in \mathbb{R}$, So, $-1 \in E_4$

d) Proposition 1

- Every upper bounded non-empty subset of $\mathbb{N}$ admits a maximum and a minimum.
- Every upper bounded non-empty subset of $\mathbb{Z}$ admits a maximum.
- Every Lower bounded non-empty subset of $\mathbb{Z}$ admits a minimum.

Proof: We will admit this proposition without demonstration.

e) Sup and inf

- Let E be an upper bounded, non-empty subset of $\mathbb{R}$.

The supremum of E (denoted $\sup E$) is the smallest (or least) upper bound of E, that is to say that:

$$M = \sup E \iff \begin{cases} M \text{ is an upper bound of } E \\ \forall M' \in M(E),\ M \leq M' \end{cases}$$

- Let E be a Lower bounded, non empty subset of $\mathbb{R}$

The infimum of E (denoted $\inf E$) is the greatest lower bound of E. This means that:

$$m = \inf E \iff \begin{cases} m \text{ is a lower bound of } E \\ \forall m' \in m(E),\ m' \leq m \end{cases}$$

Examples Consider again $E_1, \dots, E_4$.

$E_1 = \{1, 2, 3\}$ • $M(E_1) = [3, +\infty[\Rightarrow \sup E_1 = 3$

• $m(E_1) =]-\infty, 1] \Rightarrow \inf E_1 =$

$E_2 = [0, 2[$ • $M(E_2) = [2, +\infty[\Rightarrow \sup E_2 = 2$

• $m(E_2) =]-\infty, 0] \Rightarrow \inf E_2 = 0$

$E_3 =]0, +\infty[$ • $M(E_3) =]0, +\infty[\Rightarrow \sup E_3$ doesn't exist.

$m(E_3) =]-\infty, 0] \Rightarrow \inf E_3 = 0$

$E_4 = \{\cos x,\ x \in \mathbb{R}\}$ • $\sup E_4 = 1$, $\inf E_4 = 1$

f) Proposition 2

- Any upper bounded, non empty subset of $\mathbb{R}$ admits a least upper bound (suppremum). (i.e. $\sup E$ exists).
- Any Lower bounded, non empty subset of $\mathbb{R}$ admits a greatest lower bound (infimum). (i.e. $\inf E$ exists).

Proof: This proposition is accepted without proof.

Now, what are the relations between (sup and max) and (inf and min)? The answer is given in the following Proposition

g) Proposition 3

- Let $E(\neq \emptyset) \subset \mathbb{R}$ and upper-bounded.
  1/ if $\max E$ exists, then $\max E = \sup E$.
  2/ if $\sup E \in E$, then $\max E$ exists and $\max E = \sup E$
- Let $E(\neq \emptyset) \subset \mathbb{R}$ and Lower-bounded
  1/ if $\min E$ exists, then $\inf E = \min E$
  2/ if $\inf E \in E$, then $\min E$ exists and $\min E = \inf E$

Example $E_4 = \{\cos x, x \in \mathbb{R}\}$

Since we have already agreed that $\max E_4 = 1$ and $\min E_4 = -1$ then, according to proposition 3, $\sup E_4 = 1$ and $\inf E_4 = -1$

h) The sup and the inf characterizations

Important

- $M = \sup E \iff \begin{cases} M \text{ is an upper bound of } E \\ \forall \varepsilon > 0, \exists x \in E / M - \varepsilon < x \leq M \end{cases}$

Or •) $M = \sup E \iff \begin{cases} M \text{ is an upper bound of } E \\ \exists (x_n)_{n\in\mathbb{N}} \text{ a sequence of elements of } E / \lim x_n = M \end{cases}$

- $m = \inf E \iff \begin{cases} m \text{ is a lower bound of } E \\ \forall \varepsilon > 0, \exists x \in E / m \leq x < m + \varepsilon \end{cases}$

Or •) $m = \inf E \iff \begin{cases} m \text{ is a lower bound of } E. \\ \exists (x_n)_{n\in\mathbb{N}} \text{ a sequence of elements of } E / \lim x_n = m \end{cases}$

Exercises

At home

1. Find, sup, max, inf, min of the following sets, whenever they exist:

- $C = \left\{ \frac{(-1)^n}{n+1} + \frac{(-1)^n + 2}{3}, n \in \mathbb{N} \right\}$.
- $B = \left\{ \frac{n+3}{n+2}, n \in \mathbb{N} \right\}$
- $A = \left[-\frac{1}{6}, \frac{1}{2}\right[\cup \left]1, \frac{3}{2}\right]$

2. Let be $x, y \in \mathbb{R}$. Show that

$$\max(x, y) = \frac{x + y + |x - y|}{2}$$

And

$$\min(x, y) = \frac{x + y - |x - y|}{2}$$

3. $\forall x, y \in [0, 1]$,

$$\min\{xy, (1-x)(1-y)\} \leq 1$$

At the amphitheater

1. Find sup, max, inf, min of the following sets, whenever they exist:

- $A = \left\{ \frac{1}{n}, n \in \mathbb{N}^* \right\}$.
- $B = \left\{ (-1)^n + \frac{1}{n^2} \in \mathbb{N}^* \right\}$
- $C = \left\{ \sin\left(\frac{n\pi}{3}\right), n \in \mathbb{N} \right\}$

2. Show that $E = \{\sqrt{x+3}, x \in \mathbb{R}^+\}$ is not upper bounded.

3. $F = \{e^{-|n|}, n \in \mathbb{Z}\}$

Show that $\inf F = 0$.

Some more Vocabulary

الأكبر - الأعلى	The greatest	le plus grand
الأصغر - الأدنى	The least	le plus petit
متتالية	Sequence	suite
يقبل	admit	admet.
البرهان بالخلف (أو بالتناقض)	Proof by the absurd (or by contradiction)	Preuve par l'absurde (ou par la contradiction)
	Curve	Courbe
لخص - ملخص	to summarize	résumer
البرهان	Proof	Preuve
الجزء الصحيح	the integer part	la partie entière
القيمة المطلقة.	the absolute value	la valeur absolue
الأحادية.	Uniqueness	Unicité

6/ Other properties of $\mathbb{R}$

a) $\mathbb{R}$ is Archimedean

Theorem: $\forall x \in \mathbb{R}, \exists n \in \mathbb{N} / x < n$

Proof. by the absurd. (Reductio ad absurdum)

Suppose that $\exists x_0 \in \mathbb{R} / \forall n \in \mathbb{N}, x_0 \geq n$.

$\forall n \in \mathbb{N}, x_0 \geq n \Rightarrow \mathbb{N}$ is upper bounded

$\Rightarrow \sup \mathbb{N}$ exists (Since it is a non empty subset of $\mathbb{R}$), so, $\exists \alpha \in \mathbb{R} / \alpha = \sup \mathbb{N}$.

Using the characterization of the suppremum:

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} / \quad \alpha - \varepsilon < n_0 \leq \alpha$

In particular, taking $\varepsilon = 1$, we get:

$$\exists n_0 \in \mathbb{N} / \alpha - 1 < n_0 \leq \alpha$$

$$\Rightarrow \text{———} / \quad \alpha < n_0 + 1.$$

So $\left\{ \begin{array}{l} \alpha \text{ is an upper bound of } \mathbb{N} \\ n_0 + 1 \in \mathbb{N} \end{array} \right\} \Rightarrow n_0 + 1 \leq \alpha$

Summarizing: $\alpha < n_0 + 1$ and $\alpha \geq n_0 + 1$ which is absurd.

b) $\mathbb{Q}$ is dense in $\mathbb{R}$

Theorem: $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x < y \Rightarrow \exists z \in \mathbb{Q} / x < z < y$

We say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. let $x, y \in \mathbb{R}$ such that: $x < y$.

We want to find $z = \frac{q}{p}$ with $q \in \mathbb{Z}$, $p \in \mathbb{Z}^*$ and $x < z < y$.

Consider the quantity $\frac{1}{y-x}$ (>0)

$\mathbb{R}$ is archimedean, so $\exists p \in \mathbb{N}^* / p > \frac{1}{y-x} (>0)$.

$$\Rightarrow py - px > 1 \Rightarrow \boxed{py > px + 1} \cdots (1)$$

from another side, consider the set

$B = \{ n \in \mathbb{Z} / n > px \}$.

We can easily see that $B \subset \mathbb{Z}$ and that $B \neq \emptyset$ because $\mathbb{R}$ is archimedean.

B is then an nonempty lower bounded (by px) subset of $\mathbb{Z}$. It then admits a minimum.

Put $q = \min B \in \mathbb{Z}$

$$q = \min B \Rightarrow q-1 \notin B.$$

i.e. $\boxed{q-1 \leq px}$ ...(2) So,

$$\underbrace{px < q}_{\text{because } q\in B} \underbrace{\leq px+1}_{\text{by}(2)} \underbrace{< py}_{\text{by}(1)}$$

Finally: $px < q < py$

$$\Rightarrow x < \underbrace{\frac{p}{q}}_{=z} < y \Rightarrow x < z < y$$

z exists because p and q exist.

c) <u>The integer part</u>

Theorem:

$\forall x \in \mathbb{R}$, $\underbrace{\exists!}_{\text{(there exists one and only one)}} n \in \mathbb{Z}$ / $n \leq x < n+1$

<u>Note:</u> This integer n, that exists, and is unique is called the integer part of x and is often denoted $E(x)$ or $\boxed{x}$

It is the greatest integer that comes just <u>before</u> the real x

<u>Proof</u>

1/ <u>Existence:</u> •/ If $x \in \mathbb{Z}$, it is obvious that $n = x$

•/ If $x \notin \mathbb{Z}$: either $x > 0$ or $x < 0$

$\longrightarrow$ <u>$x > 0$:</u>

Consider: $A = \{m \in \mathbb{N} / m < x\}$.

$$\begin{cases} A \subset \mathbb{N} \\ A \neq \emptyset \text{ because it contains at least } 0 \\ A \text{ is upper bounded (by } x) \end{cases}$$

A is an nonempty upper bounded subset of $\mathbb{N}$, it then admits a maximum. Put $n = \max A$.

$n \in A \Rightarrow$ ① $\boxed{n < x}$, but $\boxed{x < n+1}$ (2), otherwise, we should have $x \geq n+1$ which means that

$x > n+1$ (because $x \notin \mathbb{Z}$, the equality $x = n+1$ is impossible!)

And this means that $n+1 \in A$

So $n+1 \leq n$ wich is i-possible.

Summarizing, One has $\boxed{n < x}$ --- ①

$\boxed{x < n+1}$ --- ②

Yields:

$n < x < n+1$ (Remember that $x \notin \mathbb{Z}$ in this case)

→ Now $x < 0$

$x < 0 \Rightarrow (-x) > 0$

$\Rightarrow \exists n \in \mathbb{N} / \quad n < -x < n+1$ (From the above case)

$\Rightarrow \exists n \in \mathbb{N} / \quad -n-1 < x < -n$

$\Rightarrow \exists m \in \mathbb{Z} \quad m = -n-1 / \quad m < x < m+1$

2/ Uniqueness: Let $x \in \mathbb{R}$, by the absurd, suppose that $\exists$ n and n' verifiying $n \neq n'$ and

$$\begin{cases} n \leq x < n+1 & \ldots ① \\ n' \leq x < n'+1 & \ldots ② \end{cases}$$

(∗) (∗∗)

Now, $n \neq n' \Rightarrow n > n'$ or $n < n'$

→ Case 1 : $n < n' \Rightarrow n \leq n'-1$

$\Rightarrow n+1 \leq n'$.

① $\Rightarrow n \leq x < n+1 \leq n'$

$\Rightarrow x < n'$ Contradiction with (∗)

→ Case 2, $n > n'$, by the same way, one gets,

$n' < n \Rightarrow n' \leq n-1$

$\Rightarrow n'+1 \leq n$

① $\Rightarrow n'+1 \leq n \leq x$ Contradiction with (∗∗)

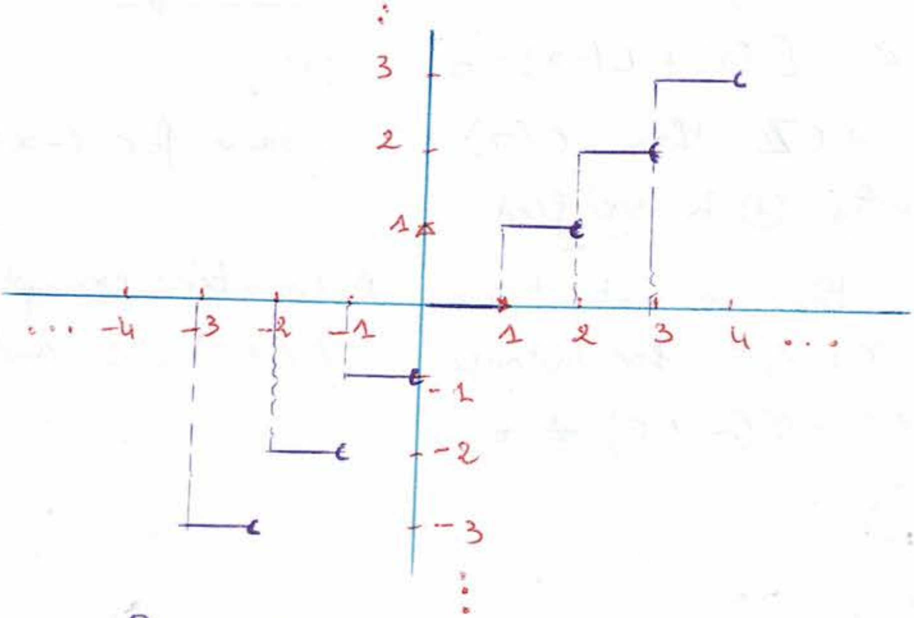
Conclusion: in both cases, we get a contradiction, this means that our supposition is false. One concluds the uniqueness of n.

Examples: $E(1{,}5) = 1$ because $1 \leq 1{,}5 < 2$

$E(-1{,}5) = -2$ because $-2 \leq -1{,}5 < -1$

$E(2) = 2$ because $2 \leq 2 < 3$.

And, in general, $\forall n \in \mathbb{Z}$, $E(n) = n$.



Representative curve of $E(x)$.

Remarks:

① We now know that

$\forall x \in \mathbb{R}, \quad \underbrace{E(x) \leq x}_{(*)} \underbrace{< E(x) + 1}_{(**)} \quad \ldots ①$

$(**) \Rightarrow x - 1 < E(x)$ and $(*) \Rightarrow E(x) \leq x$

That yields: $\forall x \in \mathbb{R}$, $x - 1 < E(x) \leq x \quad \ldots ②$

We use ① and ② very often in the practise of the integer value.

② We call the completed real line and denote by $\overline{\mathbb{R}}$ the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$

$-\infty$ 0 $+\infty$

$+\infty$ and $-\infty$ are now considered as elements of $\overline{\mathbb{R}}$

$+\infty$ is the greater element of $\overline{\mathbb{R}}$

$-\infty$ is the Lowest element of $\overline{\mathbb{R}}$

We use this notation to avoid distinguishing cases.

For example: the open intervals of $\mathbb{R}$ are,

$]a,b[$, $]a,+\infty[$, $]-\infty,b[$, $]-\infty,+\infty[$ where $a, b \in \mathbb{R}$

Using $\overline{\mathbb{R}}$, we put an open interval as $]a,b[$ where $a, b \in \overline{\mathbb{R}}$

③ All kinds of intervals: $I \subset \mathbb{R}$ is an interval $\Leftrightarrow \forall x, y \in I$, $x < z < y \Rightarrow z \in I$ $(z \in \mathbb{R})$

The interval is:	bounded	Unbounded
Open	$]a,b[$	$]a,+\infty[$, $]-\infty,a[$, $]-\infty,+\infty[$
closed	$[a,b]$	$[a,+\infty[$, $]-\infty,a]$, $\mathbb{R}$
half open or half closed	$[a,b[$, $]a,b]$	

Exercises

At home and Amphi

1. Show that,
   - $\forall x \in \mathbb{R}$, $E(x+1) = E(x) + 1$.
   - $\forall x, y \in \mathbb{R}$, $E(x) + E(y) \leq E(x+y)$
   - $\forall x, y \in \mathbb{R}$ $x \leq y \Rightarrow E(x) \leq E(y)$
2. Solve in $\mathbb{R}$ the equation
   $$\frac{1}{2} E(-(x+1)) - 1 = 3$$
3. Show that
   $\forall x \in \mathbb{Z}$ $\quad E(x) + E(-x) = 0$
   Is it true in $\mathbb{R}$ also?

Remark,

- The interection of interval is an interval.
- The union of intervals is not necessarily an interval

Example, $\quad I =]1, 2] \cup [4, 5]$.

$\exists z = 3$ (for instance) such that $\underset{\in I}{2} < z < \underset{\in I}{4}$

but $z \notin I$.

- $\emptyset$ and $\mathbb{R}$ are the only open and closed intervals at the same time of $\mathbb{R}$.