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Module: Algebra 1
L1 Math+MI

Tutorial sheet N°2 : Sets and maps

Exercise 1

(1) Write the following sets in extension:

$A = \left\{ n \in \mathbb{Z} \setminus \frac{\sqrt{2}}{2} < |n| < \sqrt{3}\,\pi \right\}$ and $B = \left\{ x \in \mathbb{R} \setminus \exists (n,p) \in \mathbb{N}^2 \ : \ x = \frac{n}{p} \text{ and } 1 \leq 2p \leq n \leq 4 \right\}$.

(2) Write the following sets in comprehension:

$C = \left\{ \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \frac{10}{11}, \frac{12}{13}, \frac{14}{15} \right\}$ and $D = \left\{ -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, -\frac{1}{7}, \ldots \right\}$.

Exercise 2

(1) Let $E = \left\{ x \in \mathbb{Z} \setminus \left| x - \frac{1}{2} \right| < 1 \right\}$ be a set. Determine $P(E)$ and $P(P(E))$.

(2) For $n \in \mathbb{N}^*$, let's pose $A_n = \left] \frac{1}{n+1}, \frac{1}{n} \right]$.

(i) Show that the family of sets $\left\{ A_n, n \in \mathbb{N}^* \right\}$ is a recovery of the interval $]0,1]$.

(ii) Deduce that $\left\{ A_n, n \in \mathbb{N}^* \right\}$ is a partition of $]0,1]$.

Exercise 3

Consider the two sets E and F defined by :

$E = \left\{ (x,y) \in \mathbb{R}^2 \setminus x^2 - xy - 2y^2 = 0 \right\}$ and $F = \left\{ (x,y) \in \mathbb{R}^2 \setminus x + y = 0 \right\}$.

(1) Show that $F \subset E$.

(2) Determine y of $\mathbb{R}$ such that : $(1,y) \in E$. Do we have $E \subset F$?

(3) Show that : $E = F \cup G$ where G is a set to be determined.

(4) Let the sets be :

$A = \left\{ (x,y) \in \mathbb{R}^2 \setminus y = x + 1 + \sqrt{x^2+1} \right\}$ and $B = \left\{ (x,y) \in \mathbb{R}^2 \setminus y = x + 1 - \sqrt{x^2+1} \right\}$.

(i) Determine the set H such that : $H = A \cup B$.

(ii) Determine $H \cap F$.

Exercise 4

Let $A, B,$ and C be three parts of a set E.

(1) Find a necessary condition so that : $A \cup B = A \cap C$

(2) Show that : $\left[A \cap B = A \cap C \text{ and } A \cup B = A \cup C \right] \Rightarrow B = C$

(3) Demonstrate that : $A \cap B = A \cup B \Rightarrow A = B$, using : (i) Direct proof. (ii) The contrapositive.

(4) $B \subset A \Rightarrow A \setminus B = C_A^B$

Exercise 5

Consider the two sets E and F defined by :

$E = \left\{ n \in \mathbb{N} \setminus n \text{ is even and divides } 24 \right\}$ and $F = \left\{ n \in \mathbb{N} \setminus n \text{ is prime and } n < 9 \right\} \cup \{1, 10\}$.

Let $f : E \longrightarrow F$ be a map defined by its graph $G = \{(2,3),(4,5),(6,1),(8,5),(12,7)\}$.

(1) Check that f is indeed a map. (2) f is it one-to-one ? onto?

(3) Determine $f(6), f(\{6\}), f\left(\{ n \in E \setminus n \text{ divides } 8 \}\right)$ and $f(E)$.

(4) Determine $f^{-1}(1), f^{-1}(\{1\}), f^{-1}(\{5\}), f^{-1}\left(\{ n \in \mathbb{N} \setminus n \text{ is prime and } n < 9 \}\right)$ and $f^{-1}(F)$.

Exercise 6

Let $f : E \longrightarrow F$ be a map, $A \in P(E)$ and $B \in P(F)$.

(1) Show that : $A \subset f^{-1}(f(A))$ and $f(f^{-1}(B)) \subset B$.

(2) Show using counter examples that : $f^{-1}(f(A)) \not\subset A$ and $B \not\subset f(f^{-1}(B))$.

(3) Find conditions on f so that : $A = f^{-1}(f(A))$ and $f(f^{-1}(B)) = B$.

Exercise 7

We notice $U =]0,+\infty[\times]0,+\infty[$. Let $f : U \longrightarrow U$ be a map defined by :

$$f(x,y) = \left(xy, \frac{y}{x}\right)$$

(1) Show that f is one-to-one? $\left(\text{We can pose } t = \frac{y}{x}\right)$.

(2) Is f onto? If yes, determine f^{-1}.

Exercice 8

Let $h : \mathbb{R}^+ \longrightarrow \left[\frac{1}{4},+\infty\right[$ be a map defined by : $\forall x \in \mathbb{R}^+,\ h(x) = x + \sqrt{x} + \frac{1}{4}$.

(1) Check that : $\forall x \in \mathbb{R}^+,\ h(x) = \left(\sqrt{x} + \frac{1}{2}\right)^2$.

(2) Write the application h as the composite of two maps f and g : $h = g \circ f$.

(3) Show that f is a bijection and determine its inverse map.

(4) Show that g is a bijection and determine its inverse map.

(5) Deduce that h is a bijection of $\mathbb{R}^+$ in $\left[\frac{1}{4},+\infty\right[$, and determine its inverse map.

Exercise 9 $(SUPP)$

(I) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map defined by : $f(x) = \dfrac{1}{\sqrt{1+x^2}}$.

(1) Determine $f\left(\{x \in \mathbb{R} \backslash\ |x| = 1\}\right)$ and $f^{-1}\left(\{y \in \mathbb{R} \backslash\ y^3 = 8\}\right)$. (2) Is f one-to-one? onto? bijective?

(3) Determine $f\left(\left[1,\sqrt{3}\,\right]\right)$, $f\left(\left]-\sqrt{3},-1\right]\right)$, $f\left(\left[-1,2\sqrt{2}\,\right[\right)$, $f(\mathbb{R}^+)$, $f^{-1}(]0,1])$ and $f^{-1}\left(\left[\frac{1}{2},1\right]\right)$.

(II) Let $g = f_{\backslash\mathbb{R}^+} : \mathbb{R}^+ \longrightarrow J$ where $J = f(\mathbb{R}^+)$. $\left(g \text{ is the restriction of } f \text{ on } \mathbb{R}^+\right)$.

(1) Show that g is bijective and determine g^{-1}. (2) Determine $g^{-1}\left(\frac{1}{2}\right)$ by two methods.

(3) Calculate $g \circ g^{-1}(y)$ for $y \in J$ and $g^{-1} \circ g(x)$ for $x \in \mathbb{R}^+$.

Exercice 10 $(SUPP)$

We notice $J =]1,+\infty[$. Let f and $g : J \longrightarrow J$ be two maps difined by :

$$\forall x \in J,\ f(x) = 1 + \frac{2}{\sqrt{x}-1} \text{ and } g(x) = \left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)^2.$$

(1) Determine $f([2,4[)$ and $g^{-1}(\{9\})$.

(2) Show that f is a bijection from J into J and determine its inverse map.

(3) Check that : $\forall x \in J,\ g(x) = (f(x))^2$.

(4) Deduce that g is a bijection of from J into J and determine its inveerse map.

SUPPLEMENTARY EXERCISES

Exercise 1

(1) Consider the following sets:

$A = \left\{ \frac{5n+8}{8n-1}, n \in \mathbb{N} \right\}$ and $B = \left\{ \frac{2n+4}{2n-1}, n \in \mathbb{N} \right\}$.

(i) Does $\frac{17}{3} \in A$? $\frac{18}{15} \in A$? $\frac{43}{25} \in B$? $\frac{42}{37} \in B$?

(ii) Show that $\frac{6}{5}$ is a common element between sets A and B.

(2) Let $C = \left\{ \frac{\pi}{4} + \frac{2k\pi}{5}, k \in \mathbb{Z} \right\}$ and $D = \left\{ \frac{\pi}{2} + \frac{2k\pi}{5}, k \in \mathbb{Z} \right\}$, be two sets.

- Show that $A \cap B = \varnothing$.

Exercise 2

Let $A =]-\infty, 3]$, $B = [-2, 7[$ and $C =]-5, +\infty[$.

Determine : $A \cup B$, $B \cap C$, $C \backslash A$, $C_{\mathbb{R}}(B)$ and $A \triangle B$.

Exercise 3

Let $A = \left\{ (x,y) \in \mathbb{R}^2 \backslash\, x^2 + y^2 \leq 1 \right\}$ be a set.

(1) Do the pairs $(1,0)$ and $(0,1)$ belong to A?

(2) Show that A can not be the cartesian product of two parts of $\mathbb{R}$.

Indication : By contradiction and notice that $(1,1) \notin A$.

Exercise 4

Let $A, B,$ and C be three parts of a set E. Show that :

(1) $(A \backslash B) \backslash C = A \backslash (B \cup C)$

(2) $C_E(A \cap B) = C_E(A) \cup C_E(B)$

(3) $A \triangle B = A \cap B \Leftrightarrow A = B = \varnothing$

Exercise 5

Consider the two parts of $\mathbb{R}^2$, $E = [0,1]$ and $F = [0,2]$

(1) Draw $E \times F$ and $E \times E$.

(2) Let $f : E \longrightarrow F$ and $g : F \longrightarrow E$ be two maps.

$x \longmapsto 2 - x \qquad x \longmapsto (x-1)^2$

(3) Specify $g \circ f$ and $f \circ g$. Do we have $g \circ f = f \circ g$ and $g \circ f = g$?

(4) Determine $f^{-1}(\{0\})$ and $g^{-1}\left(\left]0, \frac{1}{2}\right[\right)$.

(5) Show that : $g \circ f$ is bijective and specify $(g \circ f)^{-1}$

Exercise 6

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map defined by : $f(x) = \frac{2x}{1+x^2}$.

(1) Determine $f(2)$ and $f\left(\frac{1}{2}\right)$. f is it injective (one-to-one)?

(2) Solve in $\mathbb{R}$: $f(x) = 2$. f is it surjective (onto)? Show that $f(\mathbb{R}) = [-1,1]$.

(3) Show that the application g defined on $[-1,1]$ in $[-1,1]$ by: $f(x) = g(x)$ is bijective and determine its inverse g^{-1}.

Exercise 7

Show that the map $f : \mathbb{R} \longrightarrow]-1,1[$ defined by : $f(x) = \frac{x}{1+|x|}$ is bijective and determine its inverse f^{-1}.

Exercise 8

Consider the following two sets :

$$A = \left\{x \in \mathbb{R} \setminus \exists\, t \in \left[0,\ 1\right] : x = t + 2\right\} \text{ and}$$
$$B = \left\{x \in \mathbb{R} \setminus \left|x - \frac{5}{2}\right| \le \frac{1}{2}\right\}.$$

(1) Write the set B as an interval $\left[a,\ b\right]$.

(2) Show that $A = B$.

Exercise 9

Let $f : \mathbb{R} \to \mathbb{R}^+$ be a map defined by :

$$\forall x \in \mathbb{R},\ f(x) = \frac{1}{1 + \sqrt{4 + x^2}}.$$

(1) Determine $f\left(\{x \in \mathbb{R} \setminus x^2 = 4\}\right)$, $f(\mathbb{R}^+)$ and $f^{-1}\left(\{y \in \mathbb{R}^+ \setminus |y| = 1\}\right)$.

(2) f is it one-to-one? onto? bijective?

(3) Let $g = f_{\setminus\mathbb{R}^+} : \mathbb{R}^+ \to J$ where $J = f(\mathbb{R}^+)$, the restriction of f on $\mathbb{R}^+$.

- Show that g is bijective and determine g^{-1}.

Exercise 10

Let f and g be two maps defined by :

$$f : \mathbb{R}\setminus\{3\} \to \mathbb{R}/\{4\},\quad x \mapsto \frac{4x+1}{x-3} \quad \text{and} \quad g : \mathbb{R}/\{4\} \to \mathbb{R}\setminus\{3\},\quad x \mapsto \frac{3x+1}{x-4}$$

(1) Show that $\forall y \in \mathbb{R}\setminus\{4\}, \dfrac{3y+1}{y-4} \neq 3$.

(2) Determine for $x \in \mathbb{R}\setminus\left\{1,\ 3\right\}$ the image of $3x$ by f and for $x \in \mathbb{R}\setminus\left\{-2,\ 2,\ 4\right\}$ the image of x^2 by g.

(3) Determine the preimages $y = 1$ by f and by g.

(4) Determine $f \circ g$ and $g \circ f$.

(5) Show that f is injective (one-to-one).

(6) f is it surjective (onto)?

(7) f is it bijective? If yes, determine f^{-1}.