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Module: Algebra 1
L1 Math

Tutorial sheet N°1 : Logic and reasoning

Exercise 1 Let P, Q and R be propositions.

(1) Using the truth table, show that the following implication is always true :

$$[(P \Rightarrow Q) \wedge (R \Rightarrow Q)] \Rightarrow [(P \vee R) \Rightarrow Q]$$

(2) Deduce that

$$\left[(P \Rightarrow Q) \wedge \left(\overline{P} \Rightarrow Q\right)\right] \Rightarrow Q$$

Application: Let $n \in \mathbb{N}^*$, show that:

$$(i) \quad n\,(n+1) \text{ is even} \qquad (ii) \quad (SUPP)\,\frac{n\,(n+1)(n+2)}{3} \in \mathbb{N}$$

(3) Show that the following implications : (A) $\quad (P \wedge Q) \Rightarrow \overline{Q}$ and (B) $\left(P \wedge \overline{Q}\right) \Rightarrow Q$ are both true if and only if P is false.

(4) $(SUPPLEMENTARY)$ Is the following proposition true? $\quad (P \wedge Q) \Rightarrow (\overline{P} \vee Q)$

(5) $(SUPPLEMENTARY)$ Are the following propositions tautologies?

$$[(P \Rightarrow Q) \wedge P] \Rightarrow Q \quad \text{and} \quad (P \vee \overline{P}) \vee Q \Rightarrow (P \wedge \overline{P}) \wedge \overline{Q}$$

Exercise 2 $\left(\textit{Connectos NAND } \left(\textit{NOT AND}\right) \textit{ and NOR } \left(\textit{NOT OR}\right)\right)$

For two propositions P and Q, we define the connectors $NAND$ $\left(NOT\ AND\right)$ and NOR $\left(NOT\ OR\right)$ by

$$P\ NAND\ Q \Leftrightarrow P \uparrow Q \Leftrightarrow \overline{P \wedge Q} \text{ and } P\ NOR\ Q \Leftrightarrow P \downarrow Q \Leftrightarrow \overline{P \vee Q}$$

(1)Draw up the truth tables of the two connectors $NAND$ and NOR.

(2) Determine $P \uparrow P, P \downarrow P, \overline{P \uparrow Q}$ and $\overline{P \downarrow Q}$

(3) Express $\overline{P}, P \vee Q, P \wedge Q$ and $P \Rightarrow Q$ using connectors $NAND \uparrow$ et $NOR \downarrow$.

Remark : The connectors $NAND \uparrow$ and $NOR \downarrow$ are important in computing and electronics.

Exercise 3

(I) Give the negation and the truth value of the following propositions (predicates):

$$P_1 : \exists\, x \in \mathbb{R}, x^2 - 2 = 0 \qquad P_2 : \exists!\, x \in \mathbb{R}, \frac{2x}{x^2+1} = \frac{4}{5}$$

$$Q_1 : \forall\, n \in \mathbb{N}, \frac{n+1}{2} \in \mathbb{N} \qquad Q_2 : \exists\, x \in [0, \pi], \cos x + \sin x = 1$$

$$R_1 : \forall\, x \in \mathbb{R}, \exists\, y \in \mathbb{R} : y^2 = x \qquad R_2 : \exists\, x \in \mathbb{R}, \forall\, y \in \mathbb{R} : y^2 = x$$

(II) Using quantifiers and mathematical symbols, write the following expressions:

(1) The cube of any real number is positive.

(2) Some real numbers are greater than or equal to their square.

(3) Between two real numbers, there exists at least one rational number.

(4) For any real number x, there exists a naturel number n such that, x is less than n.

(5) For all real number x, there exists a relative integer n, such that x is greater than or equal to n and less than n plus 1.

(6) There exists a unique natural number less than all others natural numbers.

Exercise 4

(I) $\left(\textit{Direct reasoning}\right)$ Prove the following assertions :

$$(1)\ \forall x \in \mathbb{R}, |x-1| \leq x^2 - x + 1$$

$$(2)\ \forall x, y \in \mathbb{R}, 0 < x \leq 2 \text{ and } 0 < y \leq 2 \Rightarrow \frac{1}{x} + \frac{1}{y} \geq 1$$

$$(3) \forall x \in \mathbb{R}^+, \frac{1}{1+\sqrt{x}} = 1 - \sqrt{x} \Rightarrow x = 0$$

(II) $(\textit{Contra}\,\text{positive}\ \textit{and contradiction})$

(1) Show that $\forall x \in \mathbb{R}, (x \neq -5 \text{ and } x \neq -8) \Rightarrow \dfrac{x+2}{x+5} \neq 2.$

(2) $(SUPP)$ Show that: $\forall x, y \in \mathbb{R}, x \neq y \Rightarrow (x+1)(y-1) \neq (x-1)(y+1).$

(3) A rectangle with an area of $170\ m^2$. Show that its lenght is greater than $13m$.

(4) $(SUPP)$ Let $n \in \mathbb{N}^*$. Prove that if n is the square of,an intege, then $2n$ is not the square of an integer..

(5) $(SUPP)$ Let $a \in]1, 2[$. Show that : $\forall x \in \mathbb{R},\ x^2 + ax + 1 \neq 0$

Exercise 5

(1) Let $l_1,\ l_2 \in \mathbb{R}$. Show that the following propositions are equivalent.

$$P:\ (\forall \varepsilon > 0, |l_1 - l_2| < \varepsilon) \Rightarrow l_1 = l_2$$

$$Q:\ (\exists \varepsilon > 0, |l_1 - l_2| \geq \varepsilon)\ OU\ l_1 = l_2 0$$

$$R:\ l_1 \neq l_2 \Rightarrow (\exists \varepsilon > 0, |l_1 - l_2| \geq \varepsilon)$$

(2) Show that the proposition R is true. Deduce that P and Q are true.

$\Big($Indication : $(l_1 \neq l_2) \Rightarrow |\, l_1 - l_2| > 0\Big)$

Exercise 6 (Reasoning by induction)

(1) Show by induction that : $\forall x > 0, \forall\ n \in \mathbb{N}^*,\ (1+x)^n \geq 1 + n.x$ (Bernoulli inequality)

$(2) \forall n \in \mathbb{N}^*$, we consider the sum $S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4}.+\ldots+\dfrac{1}{n.(n+1)}$

(i) Calculate $S_1,\ S_2,\ S_3$ and S_4.

(ii) Propose or conjecture a formula in n for S_n. (iii) Prove this formula by induction.

Exercise 7 (different types of reasoning for an obvious statement)

Suggest the following statement various demonstrations : $P(n)\ : \forall n \in \mathbb{N}$, the integer $5^n + 1$ is even.

(i) Proof by induction (ii) Proof by contradiction (iii) Using a remarkable identity to factor $5^n - 1$

(iv) Using a method distinct from the previous three.

Supplementary exercises

Exercise 1 $(The\ exclusive\ OR\ (XOR)\ denoted\ \oplus)$

For two propositions A and B, we define the exclusive OR (XOR) denoted $\oplus$:by $A \oplus B$ is true if A is true or B is true and not both true at the same time.

- Show that : :

$$A \oplus B \Leftrightarrow (A \wedge \overline{B}) \vee (\overline{A} \wedge B) \Leftrightarrow (A \vee B) \wedge (\overline{A} \vee \overline{B})$$

$$A \oplus B \Leftrightarrow B \oplus A \qquad A \oplus F \Leftrightarrow A \qquad A \oplus A \Leftrightarrow F$$

where A,B,C and F are propositions with F is false.

Exercise 2 $(Three - value\ logic)$

If we do not accept the principle of the third excluded party (as is the case in intuitionist logic), (المنطق الحدسي), we are tempted to introduce a third truth value.

Value	1	0	$\frac{1}{2}$
interpretation	true	false	possible

- Determine the truth tables of the following propositions : $\overline{P},\ P \vee Q,\ P \wedge Q,\ P \Rightarrow Q$ and $P \Leftrightarrow Q$

Exercise 3

Show that :

(1) $\forall a,b \in \mathbb{R}, a^2+b^2 = 1 \Rightarrow |a+b| \leq \sqrt{2}$

(2) $\forall x \in \mathbb{R},\ x+\dfrac{1}{x} \geq 2.$ .....(3) $\forall x,y \in \mathbb{R}, \sqrt{x^2+1}+\sqrt{y^2+1} = 2 \Rightarrow x = y = 0$

(4) Solve in $\mathbb{R}$ equation : $\sqrt{x^2+1} = 2x.$

Exercise 4

Show that the following propositions are false.

(i) $\forall x \in [0,1],\ x^2 \geq x$ $\quad(ii)$ $\forall x,y \in \mathbb{R}, x^2+y^2 \geq x+y$ $\quad(iii)$ $\forall a,b \in \mathbb{R}, \sqrt{a^2+b^2} = a+b$

(iv) The function f defined on $\mathbb{R}$ by $: f(x) = x^2+2x,$ is neither even nor odd.

(v) $\forall\ n \in \mathbb{N},$ the integer n^2+n+11 is prime.

Exercise 5

Let n and p natural numbers with $n > p$.

Show that : $n.p$ is an even integer OR n^2-p^2 is a multiple of 8.

Exercise 6

Show by induction that

(1) $\forall\ n \in \mathbb{N},\ n^2 < 3^n.$ (Indication $: \forall\ n \geq 2,\ n^2 \geq 2n$ and $n^2 > 1$).

(2) $\forall\ n \in \mathbb{N}^*, \displaystyle\sum_{k=1}^{n}\frac{1}{k^2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}.+\ldots+\frac{1}{n^2} \leq 2-\frac{1}{n}.$

Exercise 7

Let $x_1,\ x_2,\ldots x_n \in [0,\ 1].$

Show by induction that :$\forall\ n \geq 1,\ \displaystyle\prod_{k=1}^{n}(1-x_k) \geq 1-\sum_{k=1}^{n}x_k.$

Exercise 8

Consider the following propositions:

$$P : \forall x \in \mathbb{R}^+, \sqrt{x^2+1} + x > 0,$$
$$Q : \forall x \in \mathbb{R}^{-*}, \sqrt{x^2+1} + x > 0 \text{ and}$$
$$R : \forall x \in \mathbb{R}, \sqrt{x^2+1} + x > 0$$

(1) Show by direct reasoning that P is true.
(2) Write the negation of the proposition Q.
(3)Show by the contradiction that Q is true.
(4) Deduce that R is true.

Exercise 9

(I) (1) Recall the sum $\sum_{k=1}^{n} k$.
(2) Verify : $2n^2 + 7n + 6 = (n+2)(2n+3)$.
(3) Show by induction that $\forall n \in \mathbb{N}^*$:

$$S_n = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(4) Deduce the sum $\sum_{k=1}^{n} k(k+1)$.
(II) Show by induction that : $\forall n \in \mathbb{N}$, $4^n + 6n - 1$ is divisible by 9.

Exercice 10

For $n \in \mathbb{N}^*$, we consider the sum

$$S_n = \sum_{k=1}^{n} k(k+1) = 1.2 + 2.3 + 3.4 + \ldots + n(n+1).$$

(1) Calculate S_1, S_2 et S_3 .
(2) Prove by induction that $\forall n \in \mathbb{N}^*$:

$$S_n = \frac{n(n+1)(n+2)}{3}.$$

(3) Deduce the sum $\sum_{k=1}^{n} k^2$.
$\left(\text{Indication : } \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \text{ and } \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k\right).$

Exercise 11

(1) Calculate $\left(\left(\sqrt{2}\right)^{\sqrt{2}}\right)^{\sqrt{2}}$ and $\left(\left(\sqrt{3}\right)^{\sqrt{2}}\right)^{\sqrt{2}}$
(2) Determine two irrational numbers a and b such that a^b is a rational number.

Exercise 12

We say that three natural numbers a, b and c form a pythagorean triple if they satisfy the relation : $a^2 + b^2 = c^2$.
(1) Let $n \in \mathbb{N}^*$. Show that $2n$, $n^2 - 1$ and $n^2 + 1$ form a pythagorean triplet.
(2) 20 and 21 are two natural numbers of a pytharorean triplet. Find the third.

Exercise 13 (Divisibility by 37)

Show that all natural numbers of the following types are always divisible by 37:
- identical 3-digit numbers $(n = aaa)$.
- identical 6-digit numbers $(n = aaaaaa)$.
- numbers written by juxtaposing two given digits three times $(n = ababab)$.