2 Sets and Maps
2-1 Sets

Contents

2-1-1 Concept of sets

In mathematics, we often encounter "sets", for example real numbers form a set.
The notion of set is fundamental in modern mathematics.
We use a simple definition from naive set theory.
Definition 1 : A **set** is a collection of distinct objects called **elements**.
We often denote a set by a capital letter $(A, B, C, D, E, F...)$ and its elements by a lower case $(a, b, c, x, y, z...)$.
If an object x is an element of a set E, we say that x **belongs to** E and we write $\mathbf{x} \in \mathbf{E}$.
Otherwise we note $\mathbf{x} \notin \mathbf{E}$ (x **does not belong to** E).
We define a set in one of the following ways:
$1-$ **By extension :** we give the **list of all its elements** (between two braces).
$2-$ **By comprehension :** we give a **property (relation)** characterizing its elements.

Examples

Sets defined by extension

$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ set of digits of the decimal system.

$B = \{a, b, c, d, ..., x, y, z\}$ alphabet set.

Sets defined by comprehension

$E = \{n \in \mathbb{N} : \; n \text{ is prime and } n \leq 20\} = \{2, 3, 5, 7, 11, 13, 17, 19\}$.

$F = \{x \in \mathbb{R} : \; |x| < 1\} =]-1, 1[$.

$G = \{x \in \mathbb{R} : \; x \geq 0 \text{ and } x \leq 0\} = \{0\}$, set containing a single element, called a **singleton**.

$H = \{x \in \mathbb{R} : x^2 + 1 = 0\}$. The equation $x^2 + 1 = 0$ does not admit real solutions.

The set H does not contain any elements. It is called an **empty** set, noted : $\emptyset$ or $\{\}$.

Remark

$\{0,1,2\} = \{1,2,0\} = \{0,1,2,1,0\}$.

Definition 2

If E is a **finite** set, then the number of its elements is called **cardinal**. It is noted : **card**(E) or $|E|$.

Examples :

- $E = \left\{ n \in \mathbb{N} : \sqrt{2} < n \leq 3\pi \right\} = \{2,3,4,5,6,7,8,9\}$.

Card$(E) = 8$

- Card$(\emptyset) = 0$
- If a set E is **infinite**, then card$(E) = +\infty$.

2-1-1 Concept of sets

Particularly important sets :

$\mathbb{N} = \{0, 1, 2, 3, ...\}$ set of natural numbers.

$\mathbb{N}^* = \{n \in \mathbb{N} : n \neq 0\} = \{n \in \mathbb{N} : n > 0\} = \{1, 2, 3, ...\}$.

$\mathbb{Z} = \{... - 3, -2, -1, 0, 1, 2, 3, ...\}$ set of integers (relative numbers).

$\mathbb{Z}^* = \{n \in \mathbb{Z} : n \neq 0\}$.

$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{Z}^* \right\}$ set of rational numbers.

$\mathbb{Q}^* = \{r \in \mathbb{Q},\ r \neq 0\}$.

$\mathbf{I}$: set of irrational numbers $\left(\pi,\ e,\ \sqrt{2},\ \sqrt{3},\ \ln 2, ...\right)$.

$\mathbb{R}$: set of real numbers (rational and irrational).

$\mathbb{R}^* = \{x \in \mathbb{R},\ x \neq 0\}$, $\mathbb{R}^+ = \{x \in \mathbb{R},\ x \geq 0\}$,

$\mathbb{R}^- = \{x \in \mathbb{R},\ x \leq 0\}$, $\mathbb{R}^{+*} = \{x \in \mathbb{R},\ x > 0\}$,

$\mathbb{R}^{-*} = \{x \in \mathbb{R},\ x < 0\}$.

$\mathbb{C} = \left\{a + ib, a, b \in \mathbb{R} \text{ and } i^2 = -1\right\}$ set of complex numbers.

2-1-2 Set comparison

Inclusion $\subset$

We say that a set A is **included** in a set B or A is a **part** of B or A is a **subset** of B, if every element of A is an element of B, and we note $\mathbf{A} \subset \mathbf{B}$.

- $[A \subset B] \iff [\forall x, \ \ (x \in A \Longrightarrow x \in B)]$
- $[A \not\subset B] \iff [\exists x, \ \ (x \in A \text{ and } x \notin B)]$

Examples

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.
- $A = \{x \in \mathbb{R}, \ 0 < x < 1\}$, $B = \{x \in \mathbb{R}, \ \ |x| < 1\}$, $C = \{x \in \mathbb{R}, \ 0 \leq x \leq 1\}$.

We have $A \subset B$, $A \subset C$ but $C \not\subset B$.

2-1-2 Set comparison

Properties : Let A, B and C be sets. We have
(i) $A \subset A$, (ii) $\emptyset \subset A$, (iii) $[A \subset B \text{ and } B \subset C] \Longrightarrow A \subset C$.

Proof

(i) $A \subset A$ is obvious $(\forall x, (x \in A \Longrightarrow x \in A))$.

(ii) $\emptyset \subset A$, by contradiction :

assume that $\emptyset \not\subset A$. So, $\exists x \in \emptyset$ and $x \notin A$.

This is a contradiction, because $\emptyset$ does not contain any elements.

(iii) $[A \subset B \text{ and } B \subset C] \Longrightarrow A \subset C$

Let $x \in A$.

$$\begin{aligned} x \in A &\Longrightarrow x \in B \text{ because } A \subset B \\ &\Longrightarrow x \in C \text{ because } B \subset C \\ &\Longrightarrow A \subset C. \end{aligned}$$

Equality $=$

Two sets A and B are **equal** if they have the same elements.

$[A = B] \iff [A \subset B \textbf{ and } B \subset A]$

$[A \neq B] \iff [A \not\subset B \textbf{ or } B \not\subset A]$.

Examples

$\mathbb{Z}^+ = \{|n| : \ n \in \mathbb{Z}\} = \{0, 1, 2, ...\} = \mathbb{N}$.

$A = \{x \in \mathbb{R} : |x| < 1\} =]-1, 1[= B$.

$E = \left\{x \in \mathbb{R} : \exists t \in \mathbb{R}, \ x = t^2 + 1\right\}$, $F = [1, +\infty[$.

Let's show that $E = F$. $[E = F] \iff [E \subset F \textbf{ and } F \subset E]$

(i) We show that : $E \subset F$. Let $x \in E$.

(ii) Let's prove that $F \subset E$. Let $x \in F$.

Conclusion : $[E \subset F \textbf{ and } F \subset E] \iff [E = F]$

Equality $=$

Two sets A and B are **equal** if they have the same elements.

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Examples

$\mathbb{Z}^+ = \{|n| : \ n \in \mathbb{Z}\} = \{0, 1, 2, ...\} = \mathbb{N}.$

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$E = \left\{x \in \mathbb{R} : \exists t \in \mathbb{R}, \ x = t^2 + 1\right\}, \ F = [1, +\infty[.$

Let's show that $E = F$. $[E = F] \iff [E \subset F$ **and** $F \subset E]$

(i) We show that : $E \subset F$. Let $x \in E$.

$x \in E \Longrightarrow \exists t \in \mathbb{R}, x = t^2 + 1 \geq 1$

$\Longrightarrow x \in [1, +\infty[= F \Longrightarrow E \subset F.$

(ii) Let's prove that $F \subset E$. Let $x \in F$.

$x \in F \Longrightarrow x \geq 1 \Longrightarrow \exists y \geq 0 : x = y + 1$

$\Longrightarrow \exists t \in \mathbb{R}, y = t^2$ and $x = t^2 + 1$

$\Longrightarrow x \in E \Longrightarrow F \subset E.$

Conclusion : $[E \subset F$ **and** $F \subset E] \iff [E = F]$

2-1-2 Set comparison

Remark :

$[A \subseteq B] \Longleftrightarrow [A \subset B$ **or** $A = B]$, $[A \subsetneq B] \Longleftrightarrow [A \subset B$ **and** $A \neq B]$.

Power set

The set formed of all parts of a set E is called the set of **power** of E and denoted $P(E)$.

Definition

For any set E, $P(E) = \{A : A \subseteq E\}$ and $A \in P(E) \Longleftrightarrow A \subseteq E$

Remark

As $\emptyset \subset E$ and $E \subset E$ then $P(E) \neq \emptyset$ and $\emptyset, E \in P(E)$.

Examples

$P(\emptyset) = \{\emptyset\} \longrightarrow \text{card}(P(\emptyset)) = 1 = 2^0$

$P(\{a\}) = \{\emptyset, \{a\}\} \longrightarrow \text{card}(P(\{a\})) = 2 = 2^1$.

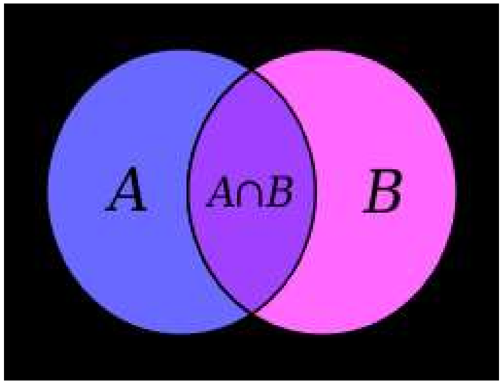
$P(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \longrightarrow \text{card}(P(\{a, b\})) = 4 = 2^2$.

$P(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
$\longrightarrow \text{card}(P(\{a, b, c\})) = 8 = 2^3$.

Proposition : If $\text{card}(E) = n$, then $\text{card}(P(E)) = 2^n$.

We now use connectives to define the set operations.
These allow us to build new set from given ones.

Definition (intersection $\cap$)
Let A and B be two parts of a set E $(A, B \in P(E))$.
The **intersection** of A and B is the set defined by
$A \cap B = \{x \in E : x \in A$ **and** $x \in B\}$. $(\cap :: \wedge)$



$x \in A \cap B \iff x \in A$ **and** $x \in B$
$x \notin A \cap B \iff x \notin A$ **or** $x \notin B$

Examples

$\mathbb{R}^+ \cap \mathbb{Z} = \mathbb{N}$

$]-1, 1] \cap [-\frac{1}{2}, 2[= [-\frac{1}{2}, 1]$.

$A = \{n \in \mathbb{N} : \ \exists k \in \mathbb{N}, \ n = 2k\}$, set of even natural numbers.

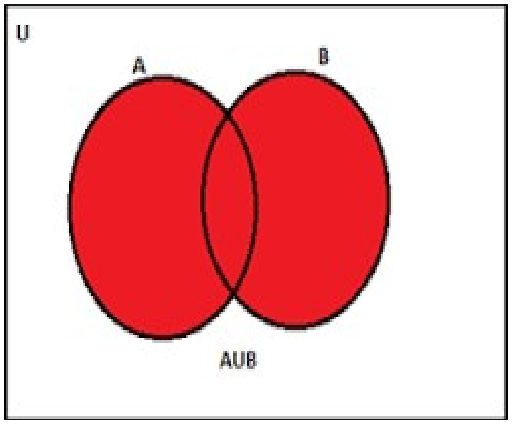
$B = \{n \in \mathbb{N} : \ \exists l \in \mathbb{N}, \ n = 2l + 1\}$, set of odd natural numbers.

$A \cap B = \emptyset$

Definition

The sets A and B are **disjoint** when $A \cap B = \emptyset$.

Definition (**union** $\cup$)
Let A and B be two parts of a set E $(A, B \in P(E))$.
The **union** of A and B is the set defined by
$A \cup B = \{x \in E : x \in A \text{ or } x \in B\}$ $(\cup :: \vee)$



$x \in A \cup B \Longleftrightarrow x \in A \text{ or } x \in B$
$x \notin A \cap B \Longleftrightarrow x \notin A \text{ and } x \notin B$

2-1-3 Set operations

Examples

$\mathbb{R}^{+} \cup \mathbb{R}^{-*} = \mathbb{R}$

$[-1, 1] \cup \left[-\frac{1}{2}, 2\right[= [-1, 2[.$

$A = \{n \in \mathbb{N} : \exists k \in \mathbb{N},\ n = 2k\}$, $B = \{n \in \mathbb{N} : \exists l \in \mathbb{N},\ n = 2l + 1\}$.

$A \cup B = \mathbb{N}$

Theorem

Let A and B be finite sets. Then we have

$(\text{card}\,(A \cup B) = \text{card}\,(A) + \text{card}\,(B) - \text{card}\,(A \cap B)\ .)$

Properties $(\cap, \cup)$

Let A, B and $C \in P(E)$.

$A \cap A = A$, $A \cap \emptyset = \emptyset$, $A \cap E = A$

$A \cup A = A$, $A \cup \emptyset = A$, $A \cup E = E$

$A \cap B = B \cap A$, $A \cup B = B \cup A$

$(A \cap B) \cap C = A \cap (B \cap C)$ and $(A \cup B) \cup C = A \cup (B \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2-1-3 Set operations

Complement of a set

Let $A, B \in P(E)$.

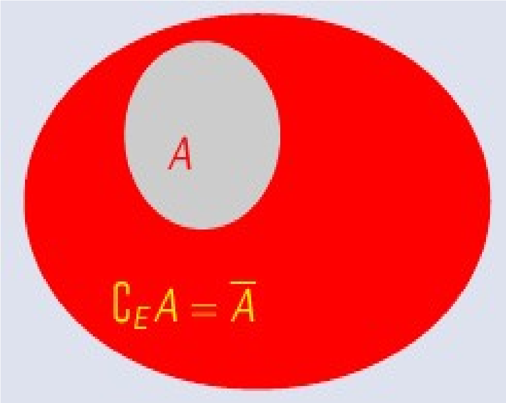
If $A \cap B = \emptyset$, we say that A and B are **disjoint**.

And if moreover $A \cup B = E$, we say that A and B are **complementary** in E.

A is the **complement** of B in E

B is the **complement** of A in E

We note by : $A = C_E^B = C_E(B)$ and $B = C_E^A = C_E(A)$



Other notation

$C_E^A = \overline{A} = A^C$ (If the set E is known)

Remark

$A = C_E^B = C_E\left(C_E^A\right) = A$

$C_E^A = \{x \in E : x \notin A\}$ $\left(C_E\left(A\right) :: \overline{A} \text{ negation}\right)$

$x \in C_E^A \Longleftrightarrow x \notin A$ and

$x \notin C_E^A \Longleftrightarrow x \in A$

Examples

$C_{\mathbb{Z}}^{\mathbb{N}} = \{-n : \ n \in \mathbb{N}^*\}$, $C_{\mathbb{C}}^{\mathbb{R}} = \{a + ib : a \in \mathbb{R} \text{ and } b \in \mathbb{R}^*\}$.

$A = \{x \in \mathbb{R} : |x| < 1\}$, $C_{\mathbb{R}}^A = \{x \in \mathbb{R} : |x| \geq 1\}$

$C_{\mathbb{R}}^{\mathbb{R}^*} = \{0\}$

$A = \left]-1, 0\right]$, $C_{\mathbb{R}}^A = \left]-\infty, -1\right] \cup \left]0, +\infty\right[$.

Properties

Let $A, B \in P(E)$.

1. $C_E^E = \emptyset$
2. $C_E^\emptyset = E$
3. $A \cap C_E^A = \emptyset$
4. $A \cup C_E^A = E$
5. $C_E\left(C_E^A\right) = A$
6. $A \subset B \Longrightarrow C_E^B \subset C_E^A$

Proof

1. $C_E^E = \emptyset$. By contradiction.

- Let's assume that $C_E^E \neq \emptyset$.
- Then $\exists x \in C_E^E \Longrightarrow x \in E$ and $x \notin E$. This is a contradiction.

2. $C_E^\emptyset = E$

$x \in C_E^\emptyset \Longleftrightarrow x \in E$ and $x \notin \emptyset \Longleftrightarrow x \in E$

3. $A \cap C_E^A = \emptyset$

By contradiction. Let's assume that $A \cap C_E^A \neq \emptyset$
Then $\exists x \in A \cap C_E^A \Longrightarrow x \in A$ and $x \in C_E^A$. This is a contradiction.
$\Longrightarrow x \in A$ and $x \notin A$. This is a contradiction.

4. $A \cup C_E^A = E$

- $x \in A \cup C_E^A \iff x \in A$ or $x \in C_E^A$
- $\iff x \in E$

5. $C_E\left(C_E^A\right) = A$

- $x \in C_E\left(C_E^A\right) \iff x \notin C_E^A \iff x \in A.$

6. $A \subset B \Longrightarrow C_E^B \subset C_E^A$

- Let $x \in C_E^B \Longrightarrow x \notin B \Longrightarrow x \notin A$ because $A \subset B$

  $\Longrightarrow x \in C_E^A$

then $C_E^B \subset C_E^A$

Finally : $A \subset B \Longrightarrow C_E^B \subset C_E^A$

De Morgan's laws

Let $A, B \in P(E)$. We have

1. $C_E(A \cap B) = C_E^A \cup C_E^B$ and
2. $C_E(A \cup B) = C_E^A \cap C_E^B$

Proof

1. $x \in C_E(A \cap B) \Longleftrightarrow x \in E \text{ and } x \notin A \cap B$

$$\Longleftrightarrow x \in E \text{ and } (x \notin A \text{ or } x \notin B).$$
$$\Longleftrightarrow (x \in E \text{ and } x \notin A) \text{ or } (x \in E \text{ and } x \notin B).$$
$$\Longleftrightarrow \left(x \in C_E^A\right) \text{ or } \left(x \in C_E^B\right).$$
$$\Longleftrightarrow x \in \left(C_E^A \cup C_E^B\right)$$

Conclusion : $C_E(A \cap B) = C_E^A \cup C_E^B$

Similary, we show by a second method that :

2. $C_E(A \cup B) = C_E^A \cap C_E^B$

$C_E(A \cup B) = C_E^A \cap C_E^B \iff$
$\left[C_E(A \cup B) \subset C_E^A \cap C_E^B \text{ and } C_E^A \cap C_E^B \subset C_E(A \cup B)\right]$
$C_E(A \cup B) \subset C_E^A \cap C_E^B$?
Let $x \in C_E(A \cup B) \Longrightarrow x \notin A \cup B \Longrightarrow x \notin A$ and $x \notin B$
$\Longrightarrow x \in C_E^A$ and $x \in C_E^B \Longrightarrow x \in C_E^A \cap C_E^B$
So, $C_E(A \cup B) \subset C_E^A \cap C_E^B$
$C_E^A \cap C_E^B \subset C_E(A \cup B)$?
Let $x \in C_E^A \cap C_E^B \Longrightarrow x \in C_E^A$ and $x \in C_E^B \Longrightarrow x \notin A$ and $x \notin B$
$\Longrightarrow x \notin A \cup B \Longrightarrow x \in C_E(A \cup B)$
So, $C_E^A \cap C_E^B \subset C_E(A \cup B)$
Conclusion : $C_E(A \cup B) = C_E^A \cap C_E^B$

Examples

Check De Morgan's laws for the following sets :
$A =]-1, 1]$, $B = [0, 2[$ and $E = \mathbb{R}$

2-1-3 Set operations

Difference of two sets

Definition

Let $A, B \in P(E)$.

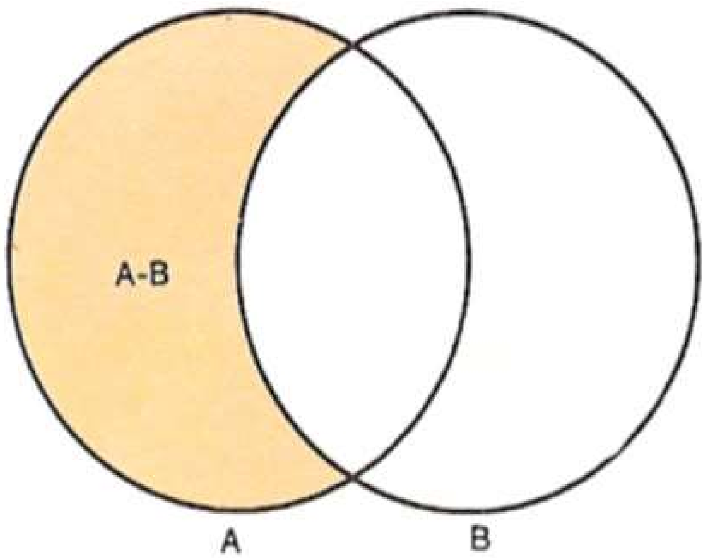
The set **difference** of B from A is the set defined by :

$A \backslash B = \{x \in E : x \in A \text{ and } x \notin B\}$.

$x \in A \backslash B \Longleftrightarrow x \in A \text{ and } x \notin B$

$x \notin A \backslash B \Longleftrightarrow x \notin A \text{ or } x \in B$

$[A \backslash B \Longleftrightarrow A - B]$



2-1-3 Set operations

Examples

$A = \,]-1, 1[\,,\ B = [0, 2]$

$A \backslash B = \,]-1, 0[\,,\ B \backslash A = [1, 2]$

$\mathbb{R} \backslash A = \,]-\infty, -1] \cup [1, +\infty[\, = C_{\mathbb{R}}^{A}$

$\mathbb{R} \backslash B = \,]-\infty, 0[\, \cup \,]2, +\infty[\, = C_{\mathbb{R}}^{B}$

$A \cap C_{\mathbb{R}}^{B} = \,]-1, 0[\, = A \backslash B$

2-1-3 Set operations

Properties

Let $A, B, C \in P(E)$.

1. $A \backslash B = A \cap C_E^B$
2. $E \backslash A = C_E^A$
3. $A \backslash A = \emptyset$
4. $A \backslash \emptyset = A$
5. $B \subset A \Longrightarrow A \backslash B = C_A^B$
6. $A \backslash (B \cup C) = (A \backslash B) \backslash C$
7. $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$

Proof

1. $A\backslash B=\{x\in E:x\in A \text{ and } x\notin B\}=\{x\in E:x\in A \text{ and } x\in C_E^B\}$

$=\{x\in E:x\in A\cap C_E^B\}=A\cap C_E^B$

2. $E\backslash A=\{x\in E:x\in E \text{ and } x\notin A\}=\{x\in E:x\notin A\}=C_E^A.$
3. $A\backslash A=\{x\in E:x\in A \text{ and } x\notin A\}=\varnothing$
4. $A\backslash\varnothing=\{x\in E:x\in A \text{ and } x\notin\varnothing\}=A$
5. Assume that $B\subset A$, and we show that $A\backslash B=C_A^B$

$A\backslash B=A\cap C_E^B=A\cap\left(C_A^B\cup C_E^A\right)=\left(A\cap C_A^B\right)\cup\left(A\cap C_E^A\right)$
$=\left(A\cap C_A^B\right)\cup\varnothing=A\cap C_A^B=C_A^B$ car $B\subset A$.

6. $A\backslash\left(B\cup C\right)=A\cap C_E\left(B\cup C\right)=A\cap\left(C_E^B\cap C_E^C\right)$

$=\left(A\cap C_E^B\right)\cap C_E^C=\left(A\backslash B\right)\cap C_E^C=\left(A\backslash B\right)\backslash C$

7. $A\backslash\left(B\cap C\right)=A\cap C_E\left(B\cap C\right)=A\cap\left(C_E^B\cup C_E^C\right)$

$=\left(A\cap C_E^B\right)\cup\left(A\cap C_E^C\right)=\left(A\backslash B\right)\cup\left(A\backslash C\right).$

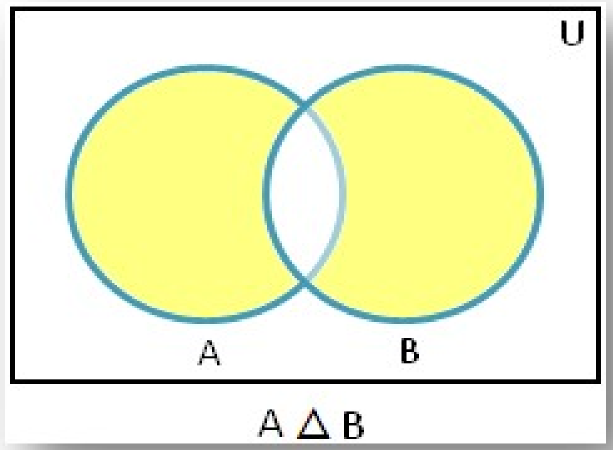
Symmetric difference of two sets

Let $A, B \in P(E)$.

The **symmetric difference** of two subsets A and B is a sub set of E, denoted by $A \triangle B$ and is defined by

$A \triangle B = \{x \in E : x \in A \backslash B \text{ or } x \in B \backslash A\}$.

$A \triangle B = (A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$



2-1-3 Set operations

Example

$E = \{n \in \mathbb{N} : 1 \leq n \leq 18\}$.

$A = \{n \in E : n \text{ is a multiple of } 3\} = \{3, 6, 9, 12, 15, 18\}$

$B = \{n \in E : n \text{ is even}\} = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$

$A \backslash B = \{3, 9, 15\}$, $B \backslash A = \{2, 4, 8, 10, 14, 16\}$

$(A \backslash B) \cup (B \backslash A) = \{2, 3, 4, 8, 9, 10, 14, 15, 16\}$

$A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18\}$

$A \cap B = \{6, 12, 18\}$

$(A \cup B) \setminus (A \cap B) = \{2, 3, 4, 8, 9, 10, 14, 15, 16\}$

We have : $A \triangle B = (A \backslash B) \cup (B \backslash A) = (A \cup B) \setminus (A \cap B)$

Properties

1. $A \triangle B = (A \backslash B) \cup (B \backslash A) = (A \cup B) \setminus (A \cap B) = (A \cup B) \cap \left(C_E^A \cup C_E^B\right)$
2. $A \triangle A = \emptyset$,
3. $A \triangle \emptyset = A$
4. $A \triangle C_E^A = E$
5. $C_E \left(A \triangle B\right) = \left(C_E^A \cup B\right) \cap \left(C_E^B \cup A\right)$

Proof

1. $A\triangle B=(A\backslash B)\cup(B\backslash A)=\{x\in E : x\in A\backslash B \text{ or } x\in B\backslash A\}.$

$=\{x\in E:(x\in A \text{ and } x\notin B) \text{ or } (x\in B \text{ and } x\notin A)\}.$
$=\{x\in E:(x\in A \text{ or } x\in B) \text{ and } (x\in A \text{ or } x\notin A) \text{ and }$
$(x\notin B \text{ or } x\in B\ ) \text{ and } (x\notin B \text{ or } x\notin A\ )\}.$
$=\{x\in E:(x\in A\cup B) \text{ and } (x\notin A\cap B)\}=(A\cup B)\setminus(A\cap B)$
$=(A\cup B)\cap C_E\,(A\cap B)=(A\cup B)\cap\left(C_E^A\cup C_E^B\right)$

2. $A\triangle A=\{x\in E : x\in A\backslash A \text{ or } x\in A\backslash A\}=\emptyset$
3. $A\triangle\emptyset=\{x\in E : x\in A\backslash\ \emptyset \text{ or } x\in\emptyset\backslash A\}=A$
4. $A\triangle C_E^A=\{x\in E : x\in A\backslash\ C_E^A \text{ or } x\in C_E^A\backslash A\}=E$
5. $C_E\,(A\triangle B)=C_E\left[(A\cup B)\cap\left(C_E^A\cup C_E^B\right)\right]$

$=C_E\,(A\cup B)\cup C_E\left[C_E^A\cup C_E^B\right]$
$=\left(C_E^A\cap C_E^B\right)\cup\left[C_E\left(C_E^A\right)\cap C_E\left(C_E^B\right)\right]$
$=\left(C_E^A\cap C_E^B\right)\cup(A\cap B)=\left(C_E^A\cup B\right)\cap\left(C_E^B\cup A\right)$

Concept of recovery and partition

Let $I = \{1, 2, 3, ...n\} \subset \mathbb{N}$, and $(A_i)_{i \in I}$ is a family of parts of a set E.

$$A = \bigcup_{i \in I} A_i = \bigcup_{i=1}^{i=n} A_i = \{x \in E : \exists i \in I,\ x \in A_i\}$$

$$x \in \bigcup_{i \in I} A_i \Longleftrightarrow \exists i \in I, x \in A_i$$

$$B = \bigcap_{i \in I} A_i = \bigcap_{i=1}^{i=n} A_i = \{x \in E :\ \forall i \in I,\ x \in A_i\}$$

$$x \in \bigcap_{i \in I} A_i \Longleftrightarrow \forall i \in I, x \in A_i$$

De Morgan's laws

1. $C_E\left(\bigcup_{i \in I} A_i\right) = \bigcap_{i \in I} C_E(A_i)$
2. $C_E\left(\bigcap_{i \in I} A_i\right) = \bigcup_{i \in I} C_E(A_i)$

Concept of recovery and partition

Definition

We say that the family $(A_i)_{i\in I}$ of parts of E is a recovery of E if and only if $\bigcup_{i\in I} A_i = E$

Example : Let $A_n =]-n, n[\,,\ n \in \mathbb{N}$.

Let's show that $(A_n)_{n\in\mathbb{N}}$ is a recovery of $\mathbb{R}$.

It is clear that $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}}]-n, n[\subset \mathbb{R}$.

Let us show that $\mathbb{R} \subset \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}}]-n, n[$.

Let $x \in \mathbb{R}$. By the property of Archimedes, $\exists n \in \mathbb{N} : |x| < n$.

So, $\exists n \in \mathbb{N} : x \in]-n, n[$ .Therefore $x \in \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}}]-n, n[$.

That is $\mathbb{R} \subset \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}}]-n, n[$, then $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}}]-n, n[= \mathbb{R}$.

We deduce that $(A_n)_{n\in\mathbb{N}}$ is a recovery of $\mathbb{R}$.

2-1-4 Concept of recovery and partition

Definition

We say that the family $(A_i)_{i\in I}$ forms a partition of E if and only if

$$\begin{cases} \forall i \in I,\ A_i \neq \emptyset \\ A_i \cap A_j = \emptyset \quad \text{if } i \neq j \\ \bigcup\limits_{i\in I} A_i = E \end{cases} \Longleftrightarrow \begin{cases} \forall i \in I,\ A_i \neq \emptyset \\ A_i \cap A_j = \emptyset \quad \text{if } i \neq j \\ (A_i)_{i\in I} \ \text{ is a recovery of } E \end{cases}$$

Examples

1. $\{\mathbb{R}^+, \mathbb{R}^{-*}\}$ is a partition of $\mathbb{R}$.
2. Let $A \subset E$. $(A \neq \emptyset)$

$\{A, C_E^A\}$ is a partition of E.

3. Let $A, B \in P(E)$

$E_1 = A\backslash B,\ E_2 = B\backslash A,\ E_3 = A\cap B,\ E_4 = C_E(A\cup B)$.
$\{E_1 = A\backslash B, E_2 = B\backslash A, E_3 = A\cap B\}$ is a partition of $A\cup B$.
$\{E_1 = A\backslash B, E_2 = B\backslash A, E_3 = A\cap B, E_4 = C_E(A\cup B)\}$ is a partition of E.

2-1-5 Cartesian product

Let A and B be two sets. Given elements $a \in A$ and $b \in B$, we call (a, b) an ordered pair. In this context, a and b are called coordinates.

Definition

The Cartesian product of A and B is the set defined by
$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

Remark

$(a, b) \neq (b, a)$
$(a, b) = (a', b') \iff a = a' \text{ and } b = b'$

Examples

$A = \{a, b, c\}$, $B = \{0, 1\}$
$A \times B = \{(a, 0), (a, 1), (b, 0), (b, 1), (c, 0), (c, 1)\}$
$B \times A = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c)\}$

2-1-5 Cartesian product

$\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$
The Cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is called the Cartesian plane.
$[0, 1] \times [0, 1] = [0, 1]^2 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$
We generalize definition of an ordered pair by defining
$A \times B \times C = \{(a, b, c) : a \in A \wedge b \in B \wedge c \in C\}$,
$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{R} \text{ and } z \in \mathbb{R}\}$
$[0, 1] \times [0, 1] \times [0, 1] = [0, 1]^3 = \{(x, y, z) \in \mathbb{R}^2 : 0 \leq x, y, z \leq 1\}$
$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) is the Cartesian n-space.