1-5 Methods of proof

Contents

1. Direct proof
2. Disjunction of cases
3. Contraposition
4. Contradiction
5. By giving a counter example
6. Successive equivalences
7. Induction.

Direct proof

- We want to show that the proposition «$P \implies Q$ » is true.
- We assume that P is true and we show that then Q is true.
- This is the method you are most familiar with.

Example 1

- Let n be a natural number.
  Show that : $\quad n$ is even $\implies n^2$ is even.
- n is even $\implies \exists\ k \in \mathbb{N}$ such that $n = 2k$
  We have $n^2 = (2k)^2 = 2\left(2k^2\right) = 2l$ with $l = 2k^2 \in \mathbb{N}$.
- And consequently n^2 is even.

Example 2

- Show that : $x, y \in \left]-1,1\right[\Longrightarrow \dfrac{x+y}{1+xy} \in \left]-1,1\right[$.
- Let $x, y \in \left]-1,1\right[$.

  $$\left(\frac{x+y}{1+xy}\right)^2 - 1 = \frac{(x+y)^2-(1+xy)^2}{(1+xy)^2} = \frac{x^2+y^2-1-x^2y^2}{(1+xy)^2}$$

  $$= \frac{x^2-1+y^2\left(1-x^2\right)}{(1+xy)^2} = \frac{\left(x^2-1\right)\left(1-y^2\right)}{(1+xy)^2} < 0$$

  because $x, y \in \left]-1,1\right[$, and $x^2 < 1$, $y^2 < 1$ then

  $$\left(\frac{x+y}{1+xy}\right)^2 - 1 < 0 \Longrightarrow \left(\frac{x+y}{1+xy}\right)^2 < 1 \Longrightarrow \left|\frac{x+y}{1+xy}\right| < 1$$

- Finally : $\left(\dfrac{x+y}{1+xy}\right)^2 - 1 < 0 \Longrightarrow \dfrac{x+y}{1+xy} \in \left]-1,1\right[$.

Disjunction of cases

- If we want to check a proposition $P(x)$ for all x in a set E,

1. we show the proposition for the x in a part A of E,
2. then for the x not belonging to A.

- This is the method of disjunction of cases or case by case.

Example 1

- Let $n \in \mathbb{N}$. Show that : $n(n+1)(n+2)$ is even.
- *First case : n is even* $\exists\ k \in \mathbb{N}$ such that $n = 2k$
  $n(n+1)(n+2) = 2k(2k+1)(2k+2) = 2l$ with
  $l = k(2k+1)(2k+2) \in \mathbb{N}$
  therefore $n(n+1)(n+2)$ is even.
- *Second case : n is odd* $\exists\ k \in \mathbb{N}$ such that $n = 2k+1$
  $$\begin{aligned} n(n+1)(n+2) &= (2k+1)(2k+2)(2k+3) \\ &= 2(2k+1)(k+1)(2k+3) \\ &= 2l \text{ with } l = (2k+1)(k+1)(2k+3) \in \mathbb{N}. \end{aligned}$$
  so, $n(n+1)(n+2)$ is even.
- Conclusion : $\forall n \in \mathbb{N}$, $n(n+1)(n+2)$ is even.

Example 2

- Show that :
$$\forall x \in \mathbb{R}, |x-2| \leq x^2 - 3x + 3$$

- $$\text{If } x \geq 2, |x-2| = x-2 \leq x^2 - 3x + 3$$
$$\text{then } x^2 - 4x + 5 \geq 0 \text{ it's true because } \triangle = -4 < 0$$

- $$\text{If } x < 2, |x-2| = 2-x \leq x^2 - 3x + 3$$
$$\text{then } x^2 - 2x + 1 = (x-1)^2 \geq 0 \text{ it is true.}$$

- Conclusion :
$$\forall x \in \mathbb{R}, |x-2| \leq x^2 - 3x + 3$$

Contraposition

- Proof by contraposition is based on the following equivalence

$$(P \implies Q) \iff (\overline{Q} \implies \overline{P})$$

- So, if we want to show the proposition $P \implies Q$
- we actually show that if $\overline{Q}$ is true then $\overline{P}$ is true.

Example 1

- Let $n \in \mathbb{N}$
  Show that : n^2 is even $\Longrightarrow n$ is even.
- Assume that n is odd , and show that n^2 is odd
- n is odd $\Longrightarrow \exists\ k \in \mathbb{N}$ shuch that $n = 2k + 1$
  We have

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1 = 2l + 1$$

  with $l = 2k^2 + 2k \in \mathbb{N}$.
  And consequently n^2 is odd
- Conclusion : n^2 is even $\Longrightarrow n$ is even.

Example 2

- Let $x, y \in \mathbb{R}$.
  Show that :

$$x \neq y \text{ and } xy \neq 1 \Longrightarrow \frac{x}{x^2 + x + 1} \neq \frac{y}{y^2 + y + 1}$$

Example 2

- $\dfrac{x}{x^2+x+1} = \dfrac{y}{y^2+y+1} \Longrightarrow x\left(y^2+y+1\right) = y\left(x^2+x+1\right)$
$$\begin{aligned}
&\Longrightarrow xy^2+xy+x = yx^2+yx+y \\
&\Longrightarrow xy^2+x-yx^2-y = 0 \\
&\Longrightarrow xy\left(y-x\right)+x-y = 0 \\
&\Longrightarrow \left(x-y\right)\left(1-xy\right) = 0 \\
&\Longrightarrow x-y = 0 \text{ or } 1-xy = 0 \\
&\Longrightarrow x = y \text{ or } xy = 1
\end{aligned}$$
- Finally : $x \neq y$ and $xy \neq 1 \Longrightarrow \dfrac{x}{x^2+x+1} \neq \dfrac{y}{y^2+y+1}$

Contradiction

- Let R be a proposition. We know that $R \vee \overline{R}$ is true.
- To show that R is true, we assume that R is false, that is to say $\overline{R}$ is true and we show that we obtain a contradiction.
- If R is an implication, $R \approxeq P \Longrightarrow Q$
  We have $\overline{P \Longrightarrow Q} \Longleftrightarrow P \wedge \overline{Q}$
- Proof by contradiction to show that $P \Longrightarrow Q$ is based on the following principle :*
  we assume both that P is true and that Q is false and we look for a contradiction.
- So if P is true then Q must be true and therefore $P \Longrightarrow Q$ is true.

Example 1

- Let show by contradiction that $\sqrt{2}$ is irrational $\left(\text{not rational}\right)$.
- We assume that $\sqrt{2} \in \mathbb{Q}$.
  $\sqrt{2} \in \mathbb{Q} \Longrightarrow \sqrt{2} = \dfrac{a}{b}$ with a and b are natural numbers that are prime to each other.
  $\left(\text{the fraction} \dfrac{a}{b} \text{is irreducible}\right)$
  $$\Longrightarrow 2 = \frac{a^2}{b^2} \Longrightarrow a^2 = 2b^2 \Longrightarrow a^2 \text{ is even} \Longrightarrow a \text{ is even}$$
  so $a = 2k$ with $k \in \mathbb{N}$
  hence $a^2 = 2b^2 \Longrightarrow (2k)^2 = 2b^2 \Longrightarrow b^2 = 2k^2$
  $$\Longrightarrow b^2 \text{ is even} \Longrightarrow b \text{ is even}$$
  a and b are both even, which contradicts the hypothesis
  $\left(a \text{ and } b \text{ are prime to each other}\right)$.
- Conclusion : $\sqrt{2}$ is irrational.

Example 2

- Let $a, b > 0$.
  Show that :

$$\frac{a}{1+b} = \frac{b}{1+a} \implies a = b$$

Example 2

- Assume that $\dfrac{a}{1+b} = \dfrac{b}{1+a}$ with $a \neq b$

  $$\begin{aligned} \frac{a}{1+b} = \frac{b}{1+a} &\implies a\left(1+b\right) = b\left(1+a\right) \\ &\implies a + a^2 = b + b^2 \implies a - b + a^2 - b^2 = 0 \\ &\implies \left(a-b\right) + \left(a-b\right)\left(a+b\right) = 0 \\ &\implies \left(a-b\right)\left(1+a+b\right) = 0 \end{aligned}$$

  and as $a \neq b$, then $1 + a + b = 0$
- This is a contradiction, because $a, b > 0$.
- Conclusion : $\dfrac{a}{1+b} = \dfrac{b}{1+a} \implies a = b$

By giving a counter example

- If we want to show that a proposition of the form $\forall x \in E,\ P(x)$ is true, then for each $x \in E$, it is necessary to show that $P(x)$ is true.
- On the other hand, to show that this proposition is false, then it suffices to find $x \in E$ such that $P(x)$ is false.
- Remember the negation of $\overline{\forall x \in E, P(x)}$ is $\exists\ x \in E, \overline{P(x)}$
- Finding such an $x \in E$ is finding a counter-example to the proposition $\forall x \in E,\ P(x)$.

Example 1

- Show that the following proposition is false :
  Any positive integer is the sum of three squares.
- $\forall n \in \mathbb{N}, \exists\ a, b$ and $c \in \mathbb{N}$, $n = a^2 + b^2 + c^2$ is false.
- The squares are $0^2, 1^2, 2^2, 3^2, 4^2, ...$
- A counter example is 7.Squares less than 7 are $0, 1$ and 4.But with three of these numbers, we cannot have 7.

Example 2

- Show that the following proposition is false :
  $\forall x, y \in \mathbb{R}, \sqrt{x^2 + y^2} = x + y$.
- $\exists\ x = -1$ and $y = 1$ such that $\sqrt{(-1)^2 + 1^2} = \sqrt{2}$ and $x + y = 0$.

Successive equivalences

- If the following equivalences are true:
  $P \iff Q_1,\ Q_1 \iff Q_2,\ Q_2 \iff Q_3, \ldots Q_n \iff Q$
- We also have the true equivalence: $P \iff Q$
- We then write to simplify the writing :
  $P \iff Q_1 \iff Q_1 \iff Q_2 \iff Q_2 \iff Q_3 \ldots Q_n \iff Q$

Example

- Let $x, y \in \mathbb{R}$. Show that
$\left(\sqrt{x^2+1}+\sqrt{y^2+1}=2\right) \Longleftrightarrow (x=y=0)$
- $\sqrt{x^2+1}+\sqrt{y^2+1}=2 \Longleftrightarrow \sqrt{x^2+1}-1+\sqrt{y^2+1}-1=0$
- As $x^2 \geq 0$ and $y^2 \geq 0$, then $x^2+1 \geq 1$ and $y^2+1 \geq 1$
and consequently $\sqrt{x^2+1}-1 \geq 0$ and $\sqrt{y^2+1}-1 \geq 0$

$$
\begin{aligned}
\sqrt{x^2+1}+\sqrt{y^2+1}=2 &\Longleftrightarrow \sqrt{x^2+1}-1+\sqrt{y^2+1}-1=0 \\
&\Longleftrightarrow \sqrt{x^2+1}-1=0 \text{ and } \sqrt{y^2+1}-1=0 \\
&\Longleftrightarrow \sqrt{x^2+1}=1 \text{ and } \sqrt{y^2+1}=1 \\
&\Longleftrightarrow x^2=0 \text{ and } y^2=0 \\
&\Longleftrightarrow x=0 \text{ and } y=0
\end{aligned}
$$

- Conclusion : $\left(\sqrt{x^2+1}+\sqrt{y^2+1}=2\right) \Longleftrightarrow (x=y=0)$

Induction

- The principle of mathematical induction method allows us to show that a proposition $P(n)$, depending on $n \in \mathbb{N}$, is true $\forall n \in \mathbb{N}$ or $\forall n \geq n_0$ $(n_0 \geq 1)$
- The proof by induction takes place in three steps :
- 1 *st step* : (*Initialization*) we verify that $P(n_0)$ is true.
- 2 *nd step* : (*heredity*) induction hypothesis, we assume that $P(n)$ is btrue for $n \geq n_0$ and we then prouve that the proposition $P(n+1)$ at the next rank is true. $(P(n) \Longrightarrow P(n+1))$.
- 3 *rd* step: (*Conclusion*) Finally in the conclusion, we recall that by the principle of induction $\forall n \geq n_0$, $P(n)$ is true.

Example 1

- Show that $\forall n \in \mathbb{N} : 7^{2n} - 1$ is divisible by 12. $\left(12/7^{2n} - 1\right)$.
- Let us note by $P(n) : 7^{2n} - 1$ is divisible by 12.
- Let us show by induction that $\forall n \in \mathbb{N}, P(n)$ is true.
- 1 *st step* : (*Initialization*) For $n = 0$, we have $7^{2.0} - 1 = 1 - 1 = 0$ as 0 is a multiple of 12 $(0 = 12.0)$, then $P(0)$ is true.

Example 1

- 2 *nd step*: $(heredity)$ we assume that $P(n)$ is true for $n \geq 0$ and we show that $P(n+1)$ is true.
  That is, we assume that $7^{2n} - 1$ is divisible by 12
  $\left(7^{2n} - 1 = 12.k \text{ with } k \in \mathbb{N}\right)$
  and we show that $7^{2(n+1)} - 1$ is divisible by 12
  $\left(7^{2(n+1)} - 1 = 12.l \text{ with } l \in \mathbb{N}\right)$
- We have
  $12/7^{2n} - 1 \Longrightarrow \exists\ k \in \mathbb{N} : 7^{2n} - 1 = 12.k \Longrightarrow 7^{2n} = 12.k + 1.$
  $$\begin{aligned} 7^{2(n+1)} - 1 = 7^{2n+2} - 1 &= 7^2.7^{2n} - 1 \\ &= 49.(12.k+1) - 1 \\ &= 49.12.k + 49 - 1 = 49.12.k + 48 \\ &= 49.12.k + 12.4 = 12.(49k+4) \\ &= 12l \text{ with } l = 49k + 4 \in \mathbb{N}. \end{aligned}$$
  Which proves that $7^{2(n+1)} - 1$ is divisible by 12.
- So $P(n+1)$ is true.

Example 1

- 3 *rd step*: (*Conclusion*) By the principle of induction, we deduce that $P(n)$ is true $\forall n \in \mathbb{N}$, that is to say, $\forall n \in \mathbb{N}$:
  $7^{2n} - 1$ is divisible by 12.
- **Remark** :
- $$\begin{aligned} 7^{2n} - 1 = 12.k &\Longrightarrow 7^{2n} = 1 + 12.k \\ &\Longrightarrow 7^2.7^{2n} = 7^2.\left(1 + 12.k\right) = 7^2 + 7^2.12.k \\ &\Longrightarrow 7^{2n+2} - 1 = 7^2 + 7^2.12.k - 1 \\ &\Longrightarrow 7^{2n+2} - 1 = 48 + 7^2.12.k = 12.\left(4 + 49.k\right) = 12.l. \end{aligned}$$

Example 2

- Show by induction that
  $\forall n \in \mathbb{N}^*,\ S_n = \sum_{k=1}^{n} k = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}.$
- $1\ st\ step$: $(Initialization)$ for $n = 1$,we have
  $$S_1 = \sum_{k=1}^{1} k = 1 = \frac{1.(1+1)}{2} = 1$$
  So the property is true for $n = 1$
- $2\ nd\ step$: $(heredity)$ suppose that $S_n = \frac{n(n+1)}{2}$
  and show that $S_{n+1} = \frac{(n+1)(n+2)}{2}$

Exemple 2

- We have $S_{n+1} = \sum_{k=1}^{n+1} k = \underbrace{1+2+3+...+n}_{S_n} + (n+1)$

  therefore $S_{n+1} = S_n + (n+1) = \dfrac{n(n+1)}{2} + (n+1)$

  (induction hypothesis)

  $$= \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

  so $S_{n+1} = \dfrac{(n+1)(n+2)}{2}$
- That is to say that the property is true for $n+1$
- 3 *rd step* : (*Conclusion*) By the principle of induction, we deduce that $\forall n \in \mathbb{N}^*,\ S_n = \sum_{k=1}^{n} k = 1+2+3+...+n = \dfrac{n(n+1)}{2}$

Sum and product symbols

- $\sum$ sum : $\underbrace{a_1 + a_2 + a_3 + ... + a_n}_{n \ terms} = \sum\limits_{k=1}^{n} a_k = \sum\limits_{i=1}^{n} a_i = \sum\limits_{j=1}^{n} a_j$
- $\underbrace{a_0 + a_1 + a_2 + a_3 + ... + a_n}_{(n+1) \ termes} = \sum\limits_{k=0}^{n} a_k$
- If $\forall k = 1, 2, ...n$, we have $a_k = a$ then $\sum\limits_{k=1}^{n} a = \underbrace{a + a + a + ... + a}_{n \ terms} = n.a$
- If $a = 1$ then $\sum\limits_{k=1}^{n} 1 = n$ and $\sum\limits_{k=0}^{n} 1 = \underbrace{1 + 1 + ... + 1}_{(n+1) \ terms} = (n+1).1 = n+1$
- $\sum\limits_{k=1}^{n} (a_k + b_k) = \sum\limits_{k=1}^{n} a_k + \sum\limits_{k=1}^{n} b_k$, $\sum\limits_{k=1}^{n} \lambda a_k = \lambda \sum\limits_{k=1}^{n} a_k$, where $\lambda \in \mathbb{R}$.

Sum and product symbols

- $\prod$ product : $\underbrace{a_1.a_2.a_3....a_n}_{n \ terms} = \prod_{k=1}^{n} a_k = \underbrace{a_0.a_1.a_2.a_3....a_n}_{(n+1) \ terms} = \sum_{k=0}^{n} a_k$
- If $\forall k = 1, 2, ...n$, we have $a_k = a$ then $\prod_{k=1}^{n} a = \underbrace{a.a.a...a}_{n \ terms} = a^n$
- $\prod_{k=1}^{n} 1 = \underbrace{1.1.1...1}_{n \ terms} = 1^n = 1 = n$ and
- $\prod_{k=0}^{n} 1 = \underbrace{1.1.1...1}_{(n+1) \ termes} = 1^{n+1} = 1, \prod_{k=1}^{n} k = 1.2.3....n = n!$ (factorial n) .
- $\prod_{k=1}^{n} (a_k.b_k) = \left(\prod_{k=1}^{n} a_k\right).\left(\prod_{k=1}^{n} b_k\right), \prod_{k=0}^{n} \lambda a_k = \lambda^n \prod_{k=1}^{n} a_k$, where $\lambda \in \mathbb{R}$.