Chapter 1 — Logic and proofs

1.0 Introduction:

Mathematical logic allows the study of mathematics as a language.

Mathematical logic is essential for the statement of a proposition and the study of its truth value. So, this is the basis of all mathematical reasoning.

1.1 Proposition (statement)

A proposition is a mathematically precise statement that is either true or false, but not both.

We often note a proposition by letters P, Q, R, ...

P: True → 1 or T

P: false → 0 or F

truth table

P
1
0

or

P
T
F

- **Principle of non-contradiction**: a pro-position can not be true and false at the same time.
- **Principle of excluded third party**: a proposition is either true or false but not a third possibility.

Examples

- 374 is divisible by 11 (proposition - True)
- The natural number 4 is less than the real number π
  (Proposition - False)
- $1+\sqrt{2}$ is not a proposition, because it does not have a true value.
- $x+1>5$ is not a proposition. The true value of this statement relies on what the variable x is assigned.
  It gets a logical statement (proposition) if we choose a value for x. (It is called a propositional function or predicate)

Definition: When a proposition depends on a variable or several variables, it is called a propositional function or predicate.

Examples

- $P(x) : e^x \geq 1$ $\begin{cases} \text{True if } x \geq 0 \\ \text{False if } x < 0 \end{cases}$

- $Q(x,y)$: For all real number x, there exists a real number y such that $y > x$.

(True : For any real number x, we can choose $y = x+1$
$y = x+1 > x$.

1.2 Logical connectives or connectors (operations).

We are particulary interested in combining propositions by operators or connectors (connectives).

Definition: A compound proposition is a statement obtained by combining propositions with logical connectives (operators).

1.2.1 Negation.

The negation of a proposition P denoted by:

not(P) or $\neg P$ or $\overline{P}$.

Truth table.

P	$\overline{P}$
1	0
0	1

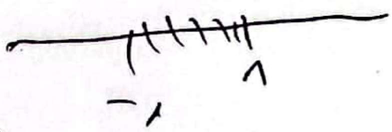
or

P	$\overline{P}$
T	F
F	T

$\boxed{\overline{(\overline{P})} \text{ is } P}$

Examples

- $P : |x| \leq 1$ - $\overline{P} : |x| > 1$



- Q : 4 is even - $\overline{Q}$: 4 is not even
  That to say: 4 is odd.
- R : All students are in the lecture hall.
  $\overline{R}$: Not all students are in the lecture hall.
  That is to say: $\overline{R}$: There is a student that is not in the lecture hall.

1.2.2 Equivalence $\Leftrightarrow$

$P \Leftrightarrow Q$ is the proposition "P is equivalent to Q" or "~~iff~~ P if and only if Q"

$P \Leftrightarrow Q$ is true when P and Q are both true or both false

True table

P	Q	$P \Leftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

Two propositions are equivalent if they have identical truth tables.

Examples

- $(a \cdot b = 0) \Leftrightarrow (a = 0 \text{ or } b = 0)$ for $a, b \in \mathbb{R}$
- n is even $\Leftrightarrow n^2$ is even

1.2.3 Conjunction "and" $\wedge$

Truth table

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

$P \wedge \overline{P}$ is always false (Principle of non-contradiction)

This time for $P \wedge Q$ to be true, we need both P and Q to be true. (False otherwise)

Examples

- (3 is prime) and (3 divides 12) $\rightarrow$ True.
  $P \quad \wedge \quad Q$
- n is an even and odd natural number $\rightarrow$ False
- $x > -1$ and $x < 1$ means $|x| < 1$

1.2.4 Disjunction "or" $\vee$

Truth table

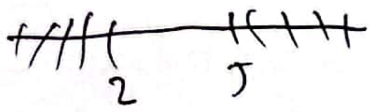
P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

$P \vee \overline{P}$ is always True (principle of excluded third party).

$P \vee Q$ is false when both P and Q are false and is true otherwise

Examples

- $\underbrace{(2 \text{ is not prime})}_{P}$ $\underbrace{\text{or}}_{\vee}$ $\underbrace{(2 \text{ divides } 5)}_{Q}$ false
- $x < -1$ or $x > 1$ means $|x| > 1$
- n is an even or odd natural number True.
- $\underbrace{x \leq 2}_{P}$ $\underbrace{\text{or}}_{\vee}$ $\underbrace{x \geq 5}_{Q}$



if $x = 1$ then $P \vee Q$ is true
if $x = 3$ then $P \vee Q$ is false

Remark Exclusive or "XOR" $\oplus$

In everyday language, there is another "OR" (exclusive)

Example: The student chooses math or computer science and not both.

True table

P	Q	$P \oplus Q$
1	1	0
1	0	1
0	1	1
0	0	0

the statement $P \oplus Q$ is true if and only if exactly one of the statements is true.

De Morgan's Laws: Negation of $\wedge$ and $\vee$

$\overline{P \wedge Q} \Leftrightarrow \overline{P} \vee \overline{Q}$ and $\overline{P \vee Q} \Leftrightarrow \overline{P} \wedge \overline{Q}$

Proof by truth table

P	Q	$\overline{P}$	$\overline{Q}$	$P \wedge Q$	$\overline{P \wedge Q}$	$\overline{P} \vee \overline{Q}$	$P \vee Q$	$\overline{P \vee Q}$	$\overline{P} \wedge \overline{Q}$
1	1	0	0	1	0	0	1	0	0
1	0	0	1	0	1	1	1	0	0
0	1	1	0	0	1	1	1	0	0
0	0	1	1	0	1	1	0	1	1

Definition: (Tautology – Antilogy (contradiction))

A proposition that is always true is called a tautology.
A proposition that is always false is called an antilogy or a contradiction.

Examples.

$P \vee \overline{P}$ is a tautology – $P \wedge \overline{P}$ is a contradiction.

1.2.5 Implication: $\Longrightarrow$ if... then...

It is an essential connective (operator) in mathematics, because it is thanks to it that mathematics advances. It allows us to state news truths.

$P \Rightarrow Q$ is the proposition "P implies Q" or "if P then Q" which is false when P is true and Q is false and true otherwise.

The mathematical definition of an implication is

$$\boxed{[P \Rightarrow Q] \Leftrightarrow [\overline{P} \vee Q]}$$

True table

P	Q	$\overline{P}$	$P \Rightarrow Q$	$\overline{P} \vee Q$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

$(P \Rightarrow Q)$ and $(\overline{P} \vee Q)$ have identical truth tables.

P is a sufficient condition for Q.
Q is a necessary condition for P.

Examples

- $1 = 2 \Rightarrow 3 = 4$ True
(because if we assume that $1=2$ then by adding 2 to both sides of this equality we obtain $3=4$)
- $0 \leq x \leq 100 \Rightarrow \sqrt{x} \leq 10$ (true) (take the square root)
- $\sin x = 0 \Rightarrow x = 0$ false
(look for $x = 2\pi$ for example)

Remark: $[P \Leftrightarrow Q] \Leftrightarrow [P \Rightarrow Q] \wedge [Q \Rightarrow P]$

P	Q	$P \Leftrightarrow Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
1	1	1	1	1	1
1	0	0	0	1	0
0	1	0	1	0	0
0	0	1	1	1	1

Negation of an implication

we know that $[P \Rightarrow Q] \Leftrightarrow [\bar{P} \vee Q]$ → definition of $\Rightarrow$

So, $\overline{P \Rightarrow Q} \Leftrightarrow \overline{\bar{P} \vee Q} \Leftrightarrow \bar{\bar{P}} \wedge \bar{Q}$ De Morgan's Laws

hence $\boxed{[\overline{P \Rightarrow Q}] \Leftrightarrow (P \wedge \bar{Q})}$

Examples:

$a, b \in \mathbb{R}$.

$R: (a=0) \text{ or } (b=0) \Rightarrow a.b=0$ true

$P \Rightarrow Q$.

$\bar{R}: [a=0 \text{ or } b=0] \text{ and } ab \neq 0$ false.

converse of an implication

$Q \Rightarrow P$ is called the converse of $P \Rightarrow Q$

$Q \Rightarrow P$ is not equivalent to $P \Rightarrow Q$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
1	1	1	1
1	0	0	1
0	1	1	0
0	0	1	1

$\boxed{[P \Rightarrow Q] \not\Leftrightarrow (Q \Rightarrow P)}$

Example: For $x \in \mathbb{R}$, $x > 5 \Rightarrow x > 1$ True

$x > 1 \Rightarrow x > 5$ false

(for example $x = 3$)

Contrapositive of an application.

$\overline{Q} \Rightarrow \overline{P}$ is called the contrapositive of $P \Rightarrow Q$

$[\overline{Q} \Rightarrow \overline{P}]$ is equivalent to $P \Rightarrow Q$.

P	Q	$\overline{P}$	$\overline{Q}$	$P \Rightarrow Q$	$\overline{Q} \Rightarrow \overline{P}$
1	1	0	0	1	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

$$[\overline{Q} \Rightarrow \overline{P}] \Leftrightarrow [P \Rightarrow Q]$$

Examples.

- $[(a=0 \text{ or } b=0) \Rightarrow a.b=0] \Leftrightarrow [ab \neq 0 \Rightarrow a \neq 0 \text{ and } b \neq 0]$
- $[n \neq 2 \text{ and } n \text{ prime} \Rightarrow n \text{ odd}] \Leftrightarrow [n \text{ even} \Rightarrow n=2 \text{ or } n \text{ not prime}]$

To remember.

$[P \Rightarrow Q] \Leftrightarrow [\overline{P} \vee Q]$ definition of $\Rightarrow$

$[\overline{P \Rightarrow Q}] \Leftrightarrow [P \wedge \overline{Q}]$ Negation of $\Rightarrow$

$[P \Rightarrow Q] \Leftrightarrow [\overline{Q} \Rightarrow \overline{P}]$ Contrapositive of $\Rightarrow$

$[Q \Rightarrow P] \not\Leftrightarrow [P \Rightarrow Q]$ converse of $P \Rightarrow Q$

$[P \Leftrightarrow Q] \Leftrightarrow [(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$

1.3 Properties of logical connectives

- $\overline{(\overline{P})} \Longleftrightarrow P$, $P \wedge Q \Longleftrightarrow Q \wedge P$, $P \vee Q \Longleftrightarrow Q \vee P$
- $P \wedge P \Longleftrightarrow P$, $P \vee P \Longleftrightarrow P$
- $(P \wedge Q) \wedge R \Longleftrightarrow P \wedge (Q \wedge R)$
- $(P \vee Q) \vee R \Longleftrightarrow P \vee (Q \vee R)$
- $P \wedge (Q \vee R) \Longleftrightarrow (P \wedge Q) \vee (P \wedge R)$
- $P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$
- $\overline{P \wedge Q} \Longleftrightarrow \overline{P} \vee \overline{Q}$, $\overline{P \vee Q} \Longleftrightarrow \overline{P} \wedge \overline{Q}$
- $[P \Rightarrow Q] \Longleftrightarrow [\overline{P} \vee Q]$, $\overline{P \Rightarrow Q} \Longleftrightarrow P \wedge \overline{Q}$
- $(P \Rightarrow Q) \Longleftrightarrow [\overline{Q} \Rightarrow \overline{P}]$
- $[P \Longleftrightarrow Q] \Longleftrightarrow [(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$
- $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Longrightarrow (P \Rightarrow R)$
- $P \wedge \overline{P}$ false $\longrightarrow$ $P \wedge F$ false where F is False
- $P \vee \overline{P}$ True $\longrightarrow$ $P \vee T$ True where T is True

1.4 Mathematical quantifiers

In mathematics, we often use expressions of the form "for any", "for all", "there exists at least", "there exists a unique."

These expressions are called quantifiers. The word quantifier comes from the word quantity.

There are two types of quantifiers.

- Universal quantifier $\forall$

$\forall x \longrightarrow$ for all x

$\forall x, P(x)$, means that the predicate $P(x)$ is true for all possible values of x.

- Existential quantifier $\exists$

$\exists x \Big\{ \longrightarrow$ there exists x (there exists at least) or there is

$\exists x, P(x)$ means that there exists an x where $P(x)$ is true

Sometimes, we will use also $\exists!, x, P(x)$

it means there exists a unique x where $P(x)$ is true

Consider the unirsal statement $\forall x, P(x)$

this asserts that $P(x)$ is true for all values of x.

Hence, if it is false, then this means that there exists an x such that $P(x)$ is false.

Similary, the existential statement $\exists x, P(x)$ asserts that there is an x where $P(x)$ is true.

Hence, if it is false, this means that for all values of x, $P(x)$ is false, that is $\overline{P(x)}$ is true

Therefore, we have the following:

$$\boxed{\begin{aligned}\overline{(\forall x, P(x))} &\iff (\exists x, \overline{P(x)})\\ \overline{\exists x, P(x)} &\iff (\forall x, \overline{P(x)})\end{aligned}}$$

Examples:

- $\forall x \in \mathbb{R}, x^2 \geq 0 \longrightarrow$ True

  Fall all real number x, the square of x is greater than or equal to zero.

- $\exists x \in \mathbb{R} \mid x^2 < 0$ false.

  There exists a real number x whose square is less than zero

- $\exists! n \in \mathbb{N} \mid n < 1.$ true $n = 0$

  There exists a unique natural number n such that n is less that one.

Remark: Some statements involve several quantifiers.

The statement: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y > x$ True

means that for all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such tath $y > x$

this statement is true

(For x $\mathbb{R}$, we can choose $y = x + 1$)

- The order of the quantifiers is very important.

The statement: $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y > x$ is false.

(There does not exist a real number which is greater tan all real numbers.

Remark :

$\forall x, \exists y, P(x,y) \not\Leftrightarrow \exists y, \forall x, P(x,y)$

$\exists x, \forall y, P(x,y) \not\Leftrightarrow \forall y, \exists x, P(x,y)$

$\forall x, \forall y, P(x,y) \Leftrightarrow \forall y, \forall x, P(x,y)$

$\exists x, \exists y, P(x,y) \Leftrightarrow \exists y, \exists x, P(x,y)$

$\overline{\forall x, \exists y, P(x,y)} \Leftrightarrow \exists x, \forall y, \overline{P(x,y)}$

$\overline{\exists x, \forall y, P(x,y)} \Leftrightarrow \forall x, \exists y, \overline{P(x,y)}$

Negation of $\exists!\, x, P(x)$

$\left(\exists!\, x \in E, P(x)\right) \Leftrightarrow \underbrace{\left[\exists x \in E, P(x)\right]}_{\text{existance}} \text{ and } \underbrace{\left[\forall x, x' \in E, P(x), P(x') \Rightarrow x = x'\right]}_{\text{uniqueness.}}$

Then

$\overline{\left[\exists!\, x \in E, P(x)\right]} \Leftrightarrow \overline{\exists x \in E, P(x)} \text{ OR } \overline{\forall x, x', P(x), P(x') \Rightarrow x = x'}$

$\overline{\left[\exists!\, x \in E, P(x)\right]} \Leftrightarrow \left[\forall x \in E, \overline{P(x)}\right] \text{ OR } \left[\exists x, x' \in E, P(x), P(x') \text{ and } x \neq x'\right]$

Example

$\exists!\, x \in \mathbb{R}, \ln x = 1 \longrightarrow$ True $x = e$ is unique.

$\overline{\exists!\, x \in \mathbb{R}, \ln x = 1} \Leftrightarrow \underbrace{\left[\forall x \in \mathbb{R}, \ln x \neq 1\right]}_{\text{false}} \text{ or } \underbrace{\left[\exists x, x' \in \mathbb{R}, \ln x = \ln x' = 1 \text{ and } x \neq x'\right]}_{\text{false.}}$