Abou Bekr Belkaid University - Tlemcen
Faculty of Sciences
Department of Mathematics

Academic year: 2023 - 2024
Module : Algebra 1
L1 Math+MI

Remedial exam

Exercise 1 (7 pts)

(1) Let $a, b > 0$.
Show that :

$$\frac{a+1}{b} = \frac{b+1}{a} \Rightarrow a = b$$

(2) Show that the following proposition is false :

$$\forall x, y \in \mathbb{R}, \sqrt{x^2 + y^2} = x + y.$$

(3) Let $x, y \in \mathbb{R}$.
Show that :

$$x \neq y \text{ and } xy \neq 1 \Rightarrow \frac{x}{x^2 + x + 1} \neq \frac{y}{y^2 + y + 1}$$

Exercise 2 (7 pts)

Let $f : \mathbb{R}^* \to \mathbb{R}$ and $g : \mathbb{R}^{*+} \to \mathbb{R}\backslash\{1\}$ be two maps.
$x \mapsto 1 - \frac{1}{x^2}$ $\quad$ $x \mapsto 1 - \frac{1}{x^2}$

(1) Determine $f(\{-1, 1\})$ and $f^{-1}(\{1\})$.
(2) f is it injective? Surjctive?
(3) Show that the map g is bijective and determine g^{-1}.

Exercise 3 (6 pts)

We define on the set of integers $\mathbb{Z}$ the relation $\Re$, called **congruence relation modulo n**, by :

$$\forall x, y \in \mathbb{Z}, (x \Re y \Leftrightarrow x \equiv y[n]) \Leftrightarrow (\exists k \in \mathbb{Z}, x - y = nk).$$

(1) Show that $\Re$ is an equivalence relation.
(2) Determine the quotient set $\mathbb{Z}/\Re$ for $n = 3$.

Good luck

Université Abou Bekr Belkaid - Tlemcen
Faculté des sciences
Département de mathématiques

Année universitaire: 2023 - 2024
Module: Algèbre 1
L1 Math+MI

Examen de rattrapage

Exercice 1 $(\mathbf{7\ pts})$

(1) Soient $a, b > 0$.
Montrer que :

$$\frac{a+1}{b} = \frac{b+1}{a} \Rightarrow a = b$$

(2) Montrer que la proposition suivante est fausse :

$$\forall x, y \in \mathbb{R}, \sqrt{x^2 + y^2} = x + y.$$

(3) Soient $x, y \in \mathbb{R}$.
Montrer que :

$$x \neq y \text{ et } xy \neq 1 \Rightarrow \frac{x}{x^2 + x + 1} \neq \frac{y}{y^2 + y + 1}$$

Exercice 2 $(\mathbf{7\ pts})$

Soient $f : \mathbb{R}^* \to \mathbb{R}$ et $g : \mathbb{R}^{*+} \to \mathbb{R} \backslash \{1\}$ deux applications.
$x \mapsto 1 - \frac{1}{x^2}$ $\quad$ $x \mapsto 1 - \frac{1}{x^2}$

(1) Déterminer $f(\{-1, 1\})$ et $f^{-1}(\{1\})$.
(2) f est-elle injective? Surjctive?
(3) Montrer que l'application g est bijective et déterminer g^{-1}.

Exercice 3 $(\mathbf{6\ pts})$

On définit sur l'ensemble des entiers $\mathbb{Z}$ la relation $\Re$, appelée relation de congruence modulo n, par :

$$\forall x, y \in \mathbb{Z}, (x \Re y \Leftrightarrow x \equiv y[n]) \Leftrightarrow (\exists k \in \mathbb{Z}, x - y = nk).$$

(1) Montrer que $\Re$ est une relation d'équivalence .
(2) Déterminer l'ensemble quotient $\mathbb{Z}/\Re$ pour $n = 3$.

Bon courage

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Exercise 1 $(\mathbf{7\ pts})$

(1) Let $a,b > 0$.

Let's show by contradiction that :

$$\frac{a+1}{b} = \frac{b+1}{a} \Rightarrow a = b$$

Assume that $\frac{a+1}{b} = \frac{b+1}{a}$ and $a \neq b$ (0,5)

$\frac{a+1}{b} = \frac{b+1}{a} \Rightarrow (a+1)a = (b+1)b$

$\Rightarrow a^2 + a = b^2 + b \Rightarrow a^2 - b^2 + a - b = 0$

$\Rightarrow (a-b)(a+b) + (a-b) = 0$ (1)

$\Rightarrow (a-b)(a+b+1) = 0$

and as $a \neq b$, then $a+b+1 = 0$

This is a contradiction, because $a,b > 0$. (1)

Conclusion : $\frac{a+1}{b} = \frac{b+1}{a} \Rightarrow a = b$

(2) Let's show by giving a counter example, that the following proposition is false :

$$\forall x,y \in \mathbb{R}, \sqrt{x^2+y^2} = x+y.$$

To show that this proposition is false, it suffices to find $x,y \in \mathbb{R}$ such that $\sqrt{x^2+y^2} = x+y$ is false. (1)

$\exists\ x = -1$ and $y = 2$ such that $\sqrt{(-1)^2+2^2} = \sqrt{5}$ and $x+y = -1+2 = 1 \neq \sqrt{5}$. (1)

(3) Let $x,y \in \mathbb{R}$.

Let's show by contraposition that :

$$x \neq y \text{ and } xy \neq 1 \Rightarrow \frac{x}{x^2+x+1} \neq \frac{y}{y^2+y+1}$$

$\frac{x}{x^2+x+1} = \frac{y}{y^2+y+1} \Rightarrow x(y^2+y+1) = y(x^2+x+1)$ (1)

$\Rightarrow xy^2 + xy + x = yx^2 + yx + y$

$\Rightarrow xy^2 + x - yx^2 - y = 0$

$\Rightarrow xy(y-x) + x - y = 0$

$\Rightarrow (x-y)(1-xy) = 0$

$\Rightarrow x - y = 0$ or $1 - xy = 0$ (1)

$\Rightarrow x = y$ or $xy = 1$

Finally : $x \neq y$ and $xy \neq 1 \Rightarrow \frac{x}{x^2+x+1} \neq \frac{y}{y^2+y+1}$ (0,5)

Exercise 2 (7 pts)

Let $f : \mathbb{R}^* \to \mathbb{R}$ and $g : \mathbb{R}^{*+} \to \mathbb{R}\backslash\{1\}$ be two maps.

$$x \mapsto 1 - \frac{1}{x^2} \qquad x \mapsto 1 - \frac{1}{x^2}$$

(1) Determine $f(\{-1,1\})$ and $f^{-1}(\{1\})$.

$$f(\{-1,1\}) = \{f(-1), f(1)\} = \left\{1 - \frac{1}{(-1)^2}, 1 - \frac{1}{(1)^2}\right\} = \{0\}$$ (1)

$f^{-1}(\{1\}) = \{x \in \mathbb{R}^* : f(x) = 1\}$

$f(x) = 1 \Leftrightarrow 1 - \frac{1}{x^2} = 1 \Leftrightarrow \frac{1}{x^2} = 0.$

There is no real x such that $\frac{1}{x^2} = 0$, so $f^{-1}(\{1\}) = \varnothing$ (1)

(2) f is it injective? Surjctive?

f is not injective, because, from (1) we have $f(-1) = f(1) = 0$ (1)

f is not surjective, because, from (1) we have $f^{-1}(\{1\}) = \varnothing$, that is there exists $y = 1 \in \mathbb{R}$ which has no preimages in $\mathbb{R}^*$. (1)

(3) Show that the map g is bijective and determine g^{-1}.

g is bijective $\Leftrightarrow$ g is injective and surjctive.

(i) Let $x_1, x_2 \in \mathbb{R}^{*+}$ such that $g(x_1) = g(x_2)$

$$g(x_1) = g(x_2) \Rightarrow 1 - \frac{1}{x_1^2} = 1 - \frac{1}{x_2^2} \Rightarrow \frac{1}{x_1^2} = \frac{1}{x_2^2}$$
$$\Rightarrow x_1^2 = x_2^2 \Rightarrow |x_1| = |x_2|$$ (1)
$$\Rightarrow x_1 = x_2, \text{ because } x_1, x_2 \in \mathbb{R}^{*+}.$$

So, g is injective.

(ii) Let $y \in \mathbb{R}\backslash\{1\}$. Let's look for an $x \in \mathbb{R}^{*+}$ such that $y = g(x)$.

$$y = g(x) \Rightarrow y = 1 - \frac{1}{x^2} \Rightarrow \frac{1}{x^2} = 1 - y \Rightarrow x^2 = \frac{1}{1-y} \text{ because } y \in \mathbb{R}\backslash\{1\}.$$
$$\Rightarrow x = \frac{1}{\sqrt{1-y}} \in \mathbb{R}^{*+}.$$

So, $\forall y \in \mathbb{R}\backslash\{1\}, \exists x = \frac{1}{\sqrt{1-y}} \in \mathbb{R}^{*+} : y = g(x)$, which implies that g is surjctive. (1)

Conclusion : as g is injective and surjective, then it is a bijection from $\mathbb{R}^{*+}$ into $\mathbb{R}\backslash\{1\}$, and its inverse g^{-1} is defined by : (1)

$$g^{-1} : \mathbb{R}\backslash\{1\} \to \mathbb{R}^{*+}$$
$$y \mapsto \frac{1}{\sqrt{1-y}}.$$

Exercise 3 (6 pts)

We define on the set of integers $\mathbb{Z}$ the relation $\mathfrak{R}$, called **congruence relation modulo n**, by :

$$\forall x, y \in \mathbb{Z}, (x \mathfrak{R} y \Leftrightarrow x \equiv y[n]) \Leftrightarrow (\exists k \in \mathbb{Z}, x - y = nk).$$

(1) Let's show that $\mathfrak{R}$ is an equivalence relation.

$\mathfrak{R}$ is an equivalence relation $\Leftrightarrow$ $\mathfrak{R}$ is reflexive, symmetric and transitive.

(i) $\mathfrak{R}$ is reflexive : $\forall x \in \mathbb{Z}, x \mathfrak{R} x$

Let $x \in \mathbb{Z}$. We have

$$x - x = 0 = 0.n \Rightarrow \exists k = 0 \in \mathbb{Z}, x - x = 0.n$$
$$\Rightarrow x \equiv x[n]$$ (1)
$$\Rightarrow x \mathfrak{R} x$$

Then $\mathfrak{R}$ is reflexive.

(ii) $\Re$ is symmetric: $\forall x,y \in \mathbb{Z},\ (x\Re y \Rightarrow y\Re x)$.
Let $x,y \in \mathbb{Z}$, such that $x\Re y$. We have

$$x\Re y \Rightarrow x \equiv y[n] \Rightarrow \exists k \in \mathbb{Z},\ x-y = k.n$$
$$\Rightarrow \exists k \in \mathbb{Z},\ -(x-y) = -(k.n)$$
$$\Rightarrow \exists k \in \mathbb{Z},\ y-x = (-k).n$$
$$\Rightarrow \exists k' = -k \in \mathbb{Z},\ y-x = k'.n$$
$$\Rightarrow y \equiv x[n]$$
$$\Rightarrow y\Re x$$

(1,5)

So, $\Re$ is symmetric.
(iii) $\Re$ is transitive: $\forall x,y,z \in \mathbb{Z},\ (x\Re y \wedge y\Re z \Rightarrow x\Re z)$.
Let $x,y,z \in \mathbb{Z}$, such that $x\Re y$ and $y\Re z$. We have

$$\begin{cases} x\Re y \\ y\Re z \end{cases} \Rightarrow \begin{cases} x \equiv y[n] \Rightarrow \exists k \in \mathbb{Z}, x-y = k.n \\ y \equiv z[n] \Rightarrow \exists k' \in \mathbb{Z}, y-z = k'.n \end{cases}$$
$$\Rightarrow \begin{cases} \exists k \in \mathbb{Z}, x-y = k.n \\ \exists k' \in \mathbb{Z}, y-z = k'.n \end{cases}$$

(1,5)

We have then $(x-y)+(y-z) = k'.n + k.n = (k+k')n$
Setting $k'' = k+k' \in \mathbb{Z}$, we have
$x - z = k''n \Rightarrow x \equiv z[n] \Rightarrow x\Re z$
Therefore $\Re$ is transitive.
As $\Re$ is reflexive, symmetric and transitive, then $\Re$ is an equivalence relation (0,5)
(2) Let's determine $\mathbb{Z}/\Re = \mathbb{Z}/n\mathbb{Z}$ for $n = 3$.
$\bar{0} = cl(0) = \{p \in \mathbb{Z} : p\Re 0\}$
$p\Re 0 \Rightarrow p \equiv 0[3] \Rightarrow \exists k \in \mathbb{Z}, p-0 = 3k \Rightarrow \exists k \in \mathbb{Z}, p = 3k$ (0,5)
Hence $\bar{0} = cl(0) = \{3k, k \in \mathbb{Z}\} = \{\ldots,-9,-6,-3,0,3,6,9,\ldots\}$
$\bar{1} = cl(1) = \{p \in \mathbb{Z} : p\Re 1\}$
$p\Re 1 \Rightarrow p \equiv 1[3] \Rightarrow \exists k \in \mathbb{Z},\ p-1 = 3k \Rightarrow \exists k \in \mathbb{Z}, p = 3k+1.$
Hence $\bar{1} = \{3k+1, k \in \mathbb{Z}\} = \{\ldots,-8,-5,-2,1,4,7,10,\ldots\}$ (0,5)
$\bar{2} = cl(2) = \{p \in \mathbb{Z} : p\Re 2\}$
$p\Re 2 \Rightarrow p \equiv 2[3] \Rightarrow \exists k \in \mathbb{Z}, p-2 = 3k \Rightarrow \exists k \in \mathbb{Z}, p = 3k+2.$ (0,5)
Hence $\bar{2} = \{3k+2, k \in \mathbb{Z}\} = \{\ldots,-7,-4,-1,2,5,8,11,\ldots\}.$
$3 \in \bar{0} \Rightarrow \bar{3} = \bar{0},\ 4 \in \bar{1} \Rightarrow \bar{4} = \bar{1},\ 5 \in \bar{2} \Rightarrow \bar{5} = \bar{2}, \ldots$ (0,5)
We notice that : $\forall x \in \mathbb{Z},\ x \in \bar{0}$ or $x \in \bar{1}$ or $x \in \bar{2}$
Hence $\mathbb{Z}/\Re = \{\bar{0},\bar{1},\bar{2}\}$
Notation $\mathbb{Z}/\Re = \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}_3$. (0,5)