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L1 Math+MI

# Final exam answer key (replacement)

**Course questions** (6 pts)

(1) Can a binary relation be both symmetric and antisymmetric?

Let $\Re$ be a binary relation defined on a set $E$ which is both symmetric and antisymmetric.
Let $x, y \in E$ be such that $x\Re y$. By symmetry, we have yRx (0,5)
So by antisymmetry, we have $x = y$. (0,5)
So: $\forall x, y \in E, x\Re y \Rightarrow x = y$.
Conclusion: The only binary relation that is both symmetric and antisymmetric is equality. (0,5)

(2) Let $\Re$ be a reflexive binary relation defined on a set $E$ such that :

$$\forall x, y, z \in E : x\Re y \text{ and } y\Re z \Rightarrow z\Re x.$$

Such a relation is called circular.
- Verify that $\Re$ is an equivalence relation.
$\Re$ is reflexive. (0,5)
Let $x, y \in E$, such that $xRy$.
We have $\forall x, y, z \in E : x\Re y$ and $y\Re z \Rightarrow z\Re x$.
For $z = y$, we have $x\Re y$ and $y\Re y \Rightarrow y\Re x$. (0,5)
So, $\Re$ is symmetric.
Let $x, y, z \in E$ such that $x\Re y$ and $y\Re z$.
$x\Re y$ and $y\Re z \Rightarrow z\Re x \Rightarrow x\Re z$ because $\Re$ is symmetric. (0,5)
Therfore $\Re$ is transitive.
Conclusion : $\Re$ is an equivalence relation. (0,5)

(3) Let $S$ be a reflexive and transitive relation defined in a set $E$ and $\triangle$ another relation defined in $E$ by :

$$\forall x, y \in E,\ x\triangle y \Leftrightarrow xSy \text{ and } ySx.$$

- Check that $\triangle$ is an equivalence relation.

We have $\forall x, y \in E,\ x\triangle y \Leftrightarrow xSy$ and $ySx$
For $y = x$, $x\triangle x \Leftrightarrow xSx$ and $xSx$, then $\triangle$ is reflexive because $S$ is reflexive. (0,5)
We have $\forall x, y \in E,\ x\triangle y \Leftrightarrow xSy$ and $ySx \Leftrightarrow ySx$ and $xSy \Leftrightarrow y\triangle x$
Then $\triangle$ is symmetric. (0,75)
Let $x, y, z \in E$ such that $x\triangle y$ and $y\triangle z$.

$$\begin{cases} x\triangle y \\ y\triangle z \end{cases} \Rightarrow \begin{cases} xSy \wedge ySx \\ ySz \wedge zSy \end{cases} \Rightarrow \begin{cases} xSy \wedge ySz \\ ySx \wedge zSy \end{cases}$$

$$\Rightarrow \begin{cases} xSy \wedge ySz \\ zSy \wedge ySx \end{cases}$$

$$\Rightarrow \begin{cases} xSz \\ zSx \end{cases} \text{ because } S \text{ is transitive.} \qquad (0{,}75)$$

$$\Rightarrow x\triangle z$$

So, $\triangle$ is transitive.
Conclusion : $\triangle$ is an equivalence relation. (0,5)

**Exercise 1** $(\mathbf{8\ pts})$

We notice $J = ]1,+\infty[$. Let $f$ and $g : J \to J$ be two maps defined by :

$$\forall x \in J,\ f(x) = 1 + \frac{2}{\sqrt{x}-1} \text{ and } g(x) = \left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)^2.$$

$(1)$ Determine $f([2,4[)$ and $g^{-1}(\{9\})$.

$$f([2,4[) = \{f(x),\ x \in [2,4[\}$$

$$x \in [2,4[ \Rightarrow 2 \le x < 4 \Rightarrow \sqrt{2} \le \sqrt{x} < 2 \Rightarrow \sqrt{2}-1 \le \sqrt{x}-1 < 1$$

$$\Rightarrow 2 < \frac{2}{\sqrt{x}-1} \le \frac{2}{\sqrt{2}-1} \qquad (0{,}5)$$

$$\Rightarrow 3 < 1 + \frac{2}{\sqrt{x}-1} \le 1 + \frac{2}{\sqrt{2}-1}$$

$$\Rightarrow 3 < f(x) \le \frac{\sqrt{2}+1}{\sqrt{2}-1} \qquad (0{,}5)$$

$$\Rightarrow 3 < f(x) \le 3 + 2\sqrt{2}$$

Then $f([2,4[) = \left[3, 3+2\sqrt{2}\right[$. (0,5)

$$g^{-1}(\{9\}) = \{x \in J,\ g(x) = 9\}$$

$$g(x) = 9 \Rightarrow \left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)^2 = 9$$

$$\Rightarrow \frac{\sqrt{x}+1}{\sqrt{x}-1} = 3 \qquad (0{,}5)$$

$$\Rightarrow \sqrt{x}+1 = 3\sqrt{x}-3$$

$$\Rightarrow \sqrt{x} = 2$$

$$\Rightarrow x = 4$$

So, $g^{-1}(\{9\}) = \{4\}$. (0,5)

$(2)$ Show that $f$ is a bijection from $J$ into $J$ and determine its inverse map.

Let's show that $f$ is injective.
Let $x_1, x_2 \in J$.

$$f(x_1) = f(x_2) \Rightarrow 1 + \frac{2}{\sqrt{x_1}-1} = 1 + \frac{2}{\sqrt{x_2}-1} \qquad (0{,}5)$$

$$\Rightarrow \sqrt{x_1} - 1 = \sqrt{x_2} - 1$$
$$\Rightarrow \sqrt{x_1} = \sqrt{x_2}$$
$$\Rightarrow x_1 = x_2 \qquad (0,5)$$
$\Rightarrow f$ is injective.

Let's show that $f$ is surjective.

Let $y \in J$, let' look for $x \in J$, such that $y = f(x)$.

$$y = f(x) \Rightarrow y = 1 + \frac{2}{\sqrt{x} - 1} \Rightarrow \frac{y-1}{2} = \frac{1}{\sqrt{x} - 1}$$
$$\Rightarrow \frac{2}{y-1} + 1 = \sqrt{x}$$
$$\Rightarrow \frac{y+1}{y-1} = \sqrt{x} \qquad (0,5)$$
$$\Rightarrow x = \left(\frac{y+1}{y-1}\right)^2$$

We have

$\left(\frac{y+1}{y-1}\right)^2 - 1 = \frac{(y+1)^2 - (y-1)^2}{(y-1)^2} = \frac{4y}{(y-1)^2} > 0$, because $y > 1$ (0,5)

Then

$\forall y \in J, \exists\, x = \left(\frac{y+1}{y-1}\right)^2 \in J$, such that $y = f(x)$

That is $f$ is surjective. (0,5)

Consequently, $f$ is bjective.

$\forall y \in J, \forall x \in J,\ y = f(x) \Leftrightarrow x = f^{-1}(y)$ (0,5)

$f^{-1} : J \to J$ difined by :

$$\forall y \in J,\ f^{-1}(y) = \left(\frac{y+1}{y-1}\right)^2 \qquad (0,5)$$

(3) Check that : $\forall x \in J,\ g(x) = (f(x))^2$.

Let $x \in J$.

$$(f(x))^2 = \left(1 + \frac{2}{\sqrt{x} - 1}\right)^2 = \left(\frac{\sqrt{x} - 1 + 2}{\sqrt{x} - 1}\right)^2 = \left(\frac{\sqrt{x} + 1}{\sqrt{x} - 1}\right)^2 = g(x). \qquad (1)$$

(4) Deduce that $g$ is a bijection from $J$ into $J$ and determine its inverse map.

We have $\forall x \in J, g(x) = (f(x))^2 = (h \circ f)(x)$, where $h(x) = x^2$ (0,5)

As $f$ and $h$ are bijective from $J$ into $J$, then $g$ is bijective from $J$ into $J$, and we have :

$g^{-1}(x) = (h \circ f)^{-1}(x) = f^{-1} \circ h^{-1}(x)$ (0,5)

We have $h^{-1}(x) = \sqrt{x}$ and $f^{-1}(x) = \left(\frac{x+1}{x-1}\right)^2$

then, $g^{-1}(x) = (h \circ f)^{-1}(x) = f^{-1} \circ h^{-1}(x) = f^{-1}(h^{-1}(x)) = \left(\frac{\sqrt{x} + 1}{\sqrt{x} - 1}\right)^2 = g(x).$

$g^{-1} = g.$ (0,5)

**Exercise 2** (**6 pts**)

(1) Show that the following proposition is false :

$\forall\ n \in \mathbb{N}$, the integer $n^2 + n + 11$ is prime.

For $n = 10$, we have $10^2 + 10 + 11 = 121 = 11^2 = 11 \times 11$ (1)

That is the integer $10^2 + 10 + 11$ is not prime.

$\exists\ n = 10 \in \mathbb{N}$, such that the integer $10^2 + 10 + 11 = 121$ is not prime. (1)

That is the proposition : $\forall\ n \in \mathbb{N}$, the integer $n^2 + n + 11$ is prime, is false.

(2) Show by induction that :

$$\forall\ n \in \mathbb{N}^*,\ \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \le 2 - \frac{1}{n}.$$

1st step: (Initialization) for $n = 1$, we have

$$\sum_{k=1}^{1} \frac{1}{k^2} = \frac{1}{1^2} = 1 \le 1 = 2 - \frac{1}{1}.$$ (0,5)

So the property is true for $n = 1$.

2nd step: (heredity )

Assume that $\sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \le 2 - \frac{1}{n}$ and show that

$$\sum_{k=1}^{n+1} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \le 2 - \frac{1}{n+1}$$ (0,5)

We have

$$\sum_{k=1}^{n+1} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} + \frac{1}{(n+1)^2} = \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2}$$ (0,5)

therefore

$$\sum_{k=1}^{n+1} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \le 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$$ (induction hypothesis) (0,5)

$$\le 2 - \frac{(n+1)^2 - n}{(n+1)^2}$$ (0,5)

$$\le 2 - \frac{n^2 + n + 1}{(n+1)^2}$$

$$\le 2 - \frac{n+1}{(n+1)^2}$$ because $-(n^2 + n + 1) \le -(n+1)$ (0,5)

So, $\sum_{k=1}^{n+1} \frac{1}{k^2} \le 2 - \frac{1}{n+1}$

That is to say that the property is true for $n + 1$. (0,5)

3rd step : (Conclusion) By the principle of induction, we deduce that :

$$\forall\ n \in \mathbb{N}^*,\ \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \le 2 - \frac{1}{n}.$$ (0,5)

Good luck