Abou Bekr Belkaid University - Tlemcen
Faculty of Sciences
Department of Mathematics

Academic year: 2023 - 2024
Module : Algebra 1
L1 Math+MI

Final exam

Exercise 1 (8 pts)

We define the maps $f : \mathbb{R} \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to [2, +\infty[$ by :

$$\forall x \in \mathbb{R},\ f(x) = 1 + \sqrt{1 + x + x^2}$$
$$\forall x \in \mathbb{R}^+,\ g(x) = 1 + \sqrt{1 + x^2}$$

(1) Find $f\left(\{x \in \mathbb{R} \setminus x^2 + x - \alpha(\alpha+1) = 0,\ \text{where } \alpha \in \mathbb{R}^{+*}\}\right)$ and $f^{-1}\left(\{y \in \mathbb{R}^+ \setminus 2|y| = 1\}\right)$.

(2) Is f injective? surjective? bijective?

(3) Show that g is bijective and determine g^{-1}.

Exercise 2 (6 pts)

Let $n \in \mathbb{N}^*$.

(1) By a direct proof, show that :

$$n \text{ is a multiple of } 3 \Rightarrow n^3 \text{ is a multiple of } 3.$$

(2) Using the contrapositive, show that :

$$n^3 \text{ is a multiple of } 3 \Rightarrow n \text{ is a multiple of } 3.$$

(3) Show by contadiction that $\sqrt[3]{3}$ is an irrational number .

Exercise 3 (6 pts)

In $\mathbb{R}$ we define the binary relations R and S by :

$$\forall x, y \in \mathbb{R},\ x\,R\,y \iff x - y \in \mathbb{Z}$$
$$\forall x, y \in \mathbb{R},\ x\,S\,y \iff x - y \in \mathbb{N}$$

(1) Show that R is an equivalence relation.

(2) Determine the equivalence classe of $\frac{1}{2}$.

(3) Show that S is an order relation.

(4) Is this order total?

Good luck

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Final exam answer key

Exercise 1 $(\mathbf{8\ pts})$

We define the maps $f:\mathbb{R}\to\mathbb{R}^+$ and $g:\mathbb{R}^+\to[2,+\infty[$ by :

$$\forall x\in\mathbb{R},\ f(x)=1+\sqrt{1+x+x^2}$$
$$\forall x\in\mathbb{R}^+,\ g(x)=1+\sqrt{1+x^2}$$

(1) Find $f\left(\{x\in\mathbb{R}\backslash\, x^2+x-\alpha(\alpha+1)=0,\ \text{where } \alpha\in\mathbb{R}^{+*}\}\right)$ and $f^{-1}\left(\{y\in\mathbb{R}^+\backslash\, 2|y|=1\}\right)$.

We put $A=\{x\in\mathbb{R}\backslash\, x^2+x-\alpha(\alpha+1)=0,\ \text{where } \alpha\in\mathbb{R}^{+*}\}$ and $B=\{y\in\mathbb{R}^+\backslash\, 2|y|=1\}$.

We have
$$\begin{aligned} x^2+x-\alpha(\alpha+1)=0 &\Leftrightarrow x^2-\alpha^2+x-\alpha=0\\ &\Leftrightarrow (x-\alpha)(x+\alpha)+x-\alpha=0\\ &\Leftrightarrow (x-\alpha)(x+\alpha+1)=0\\ &\Leftrightarrow x=\alpha \text{ or } x=-\alpha-1 \end{aligned}$$
(1)

Then $A=\{\alpha,-\alpha-1\}$

$$\begin{aligned} f(A)=\{f(\alpha),f(-\alpha-1)\}&=\left\{1+\sqrt{1+\alpha+\alpha^2},1+\sqrt{1+(-\alpha-1)+(-\alpha-1)^2}\right\}\\ &=\left\{1+\sqrt{1+\alpha+\alpha^2},1+\sqrt{1++\alpha+\alpha^2}\right\}=\left\{1+\sqrt{1+\alpha+\alpha^2}\right\}. \end{aligned}$$
(0,5)

$B=\{y\in\mathbb{R}^+\backslash\, 2|y|=1\}=\{y\in\mathbb{R}^+\backslash\, |y|=\frac{1}{2}\}=\{\frac{1}{2}\}$. (0,25)

$f^{-1}(B)=\{x\in\mathbb{R}:f(x)=\frac{3}{2}\}$

$$\begin{aligned} f(x)=\frac{1}{2} &\Rightarrow 1+\sqrt{1+x+x^2}=\frac{1}{2}\Rightarrow\sqrt{1+x+x^2}=\frac{1}{2}-1=-\frac{1}{2}\\ &\Rightarrow 1+x+x^2=\frac{1}{4}\Rightarrow x^2+x+\frac{3}{4}=0 \end{aligned}$$
(0,5)

This equation has no solution because its discriminant $\triangle=-2<0$.

So, $f^{-1}(B)=\varnothing$. (0,25)

(2) Is f injective? surjective? bijective?

f is not injective, because $\exists x_1=\alpha, x_2=-\alpha-1$, such that :
$\alpha\neq-\alpha-1$ and $f(\alpha)=f(-\alpha-1)=1+\sqrt{1+\alpha+\alpha^2}$, for $\alpha\in\mathbb{R}^{+*}$. (1)

f is not surjective, because, $\exists y=\frac{1}{2}\in\mathbb{R}^+,\forall x\in\mathbb{R}:f(x)\neq\frac{1}{2}$.
That is, there exists $y=\frac{1}{2}\in\mathbb{R}^+$, which does not have a preimage. (1)

As f is neither injective nor surjective, then it is not bijective. (0,5)

(3) Show that g is bijective and determine g^{-1}.

g is injective $\Leftrightarrow\forall x_1,x_2\in\mathbb{R}^+,(g(x_1)=g(x_2)\Rightarrow x_1=x_2)$ (0,25)

Let $x_1,x_2\in\mathbb{R}^+$.

$$\begin{aligned} g(x_1)=g(x_2) &\Rightarrow 1+\sqrt{1+x_1^2}=1+\sqrt{1+x_2^2}\\ &\Rightarrow\sqrt{1+x_1^2}=\sqrt{1+x_2^2}\Rightarrow 1+x_1^2=1+x_2^2\\ &\Rightarrow x_1^2=x_2^2\Rightarrow|x_1|=|x_2|\\ &\Rightarrow x_1=x_2,\ \text{because } x_1,x_2\in\mathbb{R}^+. \end{aligned}$$
(0,75)

Hence g is injective.

g is surjective $\Leftrightarrow \forall y \in [2,+\infty[, \exists x \in \mathbb{R}^+ : y = g(x).$ (0,5)

Let $y \in [2,+\infty[$. Let's look for an $x \in \mathbb{R}^+$, such that $y = g(x)$.

$$y = g(x) \Rightarrow y = 1+\sqrt{1+x^2} \Rightarrow y-1 = \sqrt{1+x^2}$$
$$\Rightarrow (y-1)^2 = 1+x^2$$
$$\Rightarrow x^2 = (y-1)^2 - 1 \geq 0, \text{ because } y \geq 2.$$
$$\Rightarrow x = \pm\sqrt{(y-1)^2-1} = \pm\sqrt{y^2-2y}$$

$\forall y \in [2,+\infty[, \exists x = \sqrt{y^2-2y} \in \mathbb{R}^+ : y = g(x) = 1+\sqrt{1+x^2}$. (1)

So, g is surjective.

As g is injective and sirjective, then it is bijective. (0,25)

$\forall y \in [2,+\infty[, \exists! x = \sqrt{y^2-2y} \in \mathbb{R}^+ : y = g(x) = 1+\sqrt{1+x^2}$.

Hence, g admits an inverse map $g^{-1} : [2,+\infty[\longrightarrow \mathbb{R}^+$ defined by : (0,75)

$$\forall y \in [2,+\infty[,\ g^{-1}(y) = \sqrt{y^2-2y}.$$

<u>Exercise 2</u> $(\mathbf{6\ pts})$

Let $n \in \mathbb{N}^*$.

(1) By a direct proof, show that :

$$n \text{ is a multiple of } 3 \Rightarrow n^3 \text{ is a multiple of } 3.$$

n is a multiple of $3 \Rightarrow \exists k \in \mathbb{N}^* : n = 3k$

$n^3 = (3k)^3 = 3^3k^3 = 3(3^2k^3) = 3l$ with $l = (3^2k^3) \in \mathbb{N}^*$. (1,5)

That is n^3 is a multiple of 3

(2) By using the contrapositive, show that :

$$n^3 \text{ is a multiple of } 3 \Rightarrow n \text{ is a multiple of } 3.$$

Let's show that :

$$n \text{ is not a multiple of } 3 \Rightarrow n^3 \text{ is not a multiple of } 3. \quad (0,5)$$

n is not a multiple of $3 \Rightarrow n = 3k+1$ or $n = 3k+2$, where $k \in \mathbb{N}^*$. (0,5)

If $n = 3k+1$, then $n^3 = (3k+1)^3 = 3^3k^3 + 3(3k)^2 + 3(3k) + 1^3$
$$= 3(3^2k^3 + 3^2k^2 + 3k) + 1 = 3l+1$$
with $l = (3^2k^3 + 3^2k^2 + 3k) \in \mathbb{N}^*$. (0,5)

That is n^3 is not a multiple of 3.

If $n = 3k+2$, then $n^3 = (3k+2)^3 = 3^3k^3 + 3.2.(3k)^2 + 3.2^2.(3k) + 2^3$
$$= 3^3k^3 + 3.2.(3k)^2 + 3.2^2.(3k) + 6 + 2$$
$$= 3(3^2k^3 + 2.3^2k^2 + 2^2.3k + 2) + 2 = 3l+2 \quad (0,5)$$
with $l = (3^2k^3 + 2.3^2k^2 + 2^2.3k + 2) \in \mathbb{N}^*$.

That is n^3 is not a multiple of 3.

Therefore,

$$n \text{ is not a multiple of } 3 \Rightarrow n^3 \text{ is not a multiple of } 3.$$

Consequently

$$n^3 \text{ is a multiple of } 3 \Rightarrow n \text{ is a multiple of } 3. \quad (0,5)$$

(3) Show by contadiction that $\sqrt[3]{3}$ is an irrational number .

We assume that : $\sqrt[3]{3}$ is a rational number.

$\sqrt[3]{3} \in \mathbb{Q} \Rightarrow \sqrt[3]{3} = \frac{a}{b}$ with a and b are natural numbers that are prime to each other. (0,5)

$$\Rightarrow 3 = \frac{a^3}{b^3} \Rightarrow a^3 = 3b^3 \Rightarrow a^3 \text{ is a multiple of } 3.$$
$$\Rightarrow a \text{ is a multiple of } 3. \quad (0,5)$$

So, $a = 3k$ with $k \in \mathbb{N}^*$.

Hence, $a^3 = 3b^3 \Rightarrow (3k)^3 = 3b^3 \Rightarrow 3^3k^3 = 3b^3$

$\Rightarrow b^3 = 3(3k^3)$

$\Rightarrow b^3$ is a multiple of 3. (0,5)

$\Rightarrow b$ is a multiple of 3.

a and b are both multiple of 3, which contradicts the hypothesis.

(a and b are prime to each other).

Conclusion : $\sqrt[3]{3}$ is an irrational number. (0,5)

Exercise 3 (**6 pts**)

In $\mathbb{R}$ we define the relations R and S by :

$$\forall x,\ y \in \mathbb{R},\ x\,R\,y \iff x - y \in \mathbb{Z}$$

$$\forall x,\ y \in \mathbb{R},\ x\,S\,y \iff x - y \in \mathbb{N}$$

(1) Show that R is an equivalence relation.

R is reflexive $\iff \forall x \in \mathbb{R},\ xRx.$ (0,25)

Let $x \in \mathbb{R}$.

We have $x - x = 0 \in \mathbb{Z} \Rightarrow xRx.$ (0,25)

Hence R is reflexive.

R is symmetric $\iff \forall x,y \in \mathbb{R}, (xRy \Rightarrow yRx)$. (0,25)

Let $x,y \in \mathbb{R}$.

$xRy \Rightarrow x - y \in \mathbb{Z} \Rightarrow \exists k \in \mathbb{Z} : x - y = k \Rightarrow \exists k \in \mathbb{Z} : -(x-y) = -k$

$\Rightarrow \exists k \in \mathbb{Z} : y - x = -k$

$\Rightarrow \exists k' = -k \in \mathbb{Z} : y - x = k'$

$\Rightarrow x - y \in \mathbb{Z}$ (0,5)

$\Rightarrow yRx$

So, R is symmetric.

R is transitive $\iff \forall x,y,z \in \mathbb{R}, (xRy \wedge yRz \Rightarrow xRz)$. (0,25)

Let $x,y,z \in \mathbb{R}$.

$$\begin{cases} xRy \\ yRz \end{cases} \Rightarrow \begin{cases} x - y \in \mathbb{Z} \\ y - z \in \mathbb{Z} \end{cases} \Rightarrow \begin{cases} \exists k \in \mathbb{Z} : x - y = k \\ \exists k' \in \mathbb{Z} : y - z = k' \end{cases}$$

$\Rightarrow (x-y) + (y-z) = k + k'$

$\Rightarrow x - z = k + k'$

$\Rightarrow \exists k'' = k + k' \in \mathbb{Z} : x - z = k''$ (0,5)

$\Rightarrow x - z \in \mathbb{Z}$

$\Rightarrow xRz$

So, R is transitive.

As R is reflexive, symmetric and transitive, then it is an equivalence relation. (0,25)

(2) Determine the equivalence classe of $\frac{1}{2}$.

$cl\left(\frac{1}{2}\right) = \overline{\left(\frac{1}{2}\right)} = \left\{x \in \mathbb{Z} : xR\frac{1}{2}\right\}.$ (0,5)

$xR\frac{1}{2} \Rightarrow x - \frac{1}{2} \in \mathbb{Z} \Rightarrow \exists k \in \mathbb{Z} : x - \frac{1}{2} = k$

$\Rightarrow \exists k \in \mathbb{Z} : x = \frac{1}{2} + k.$

$cl\left(\frac{1}{2}\right) = \overline{\left(\frac{1}{2}\right)} = \left\{\frac{1}{2} + k,\ k \in \mathbb{Z}\right\} = \left\{\frac{2k+1}{2},\ k \in \mathbb{Z}\right\} = \left\{\ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \frac{2n+1}{2} \ldots\right\}.$ (0,5)

(3) Show that S is an order relation.

S is reflexive $\Leftrightarrow \forall x \in \mathbb{R},\ xSx.$ (0,25)

Let $x \in \mathbb{R}$.

We have $x - x = 0 \in \mathbb{N} \Rightarrow xSx.$ (0,25)

Hence S is reflexive.

S is antisymmetric $\Leftrightarrow \forall x,y \in \mathbb{R}, (xSy \wedge ySx \Rightarrow x = y)$. (0,25)

Let $x,y \in \mathbb{R}$.

$$\begin{cases} xSy \\ ySx \end{cases} \Rightarrow \begin{cases} x-y \in \mathbb{N} \\ y-x \in \mathbb{N} \end{cases} \Rightarrow \begin{cases} \exists k \in \mathbb{N} : x-y = k \\ \exists k' \in \mathbb{N} : y-x = k' \end{cases}$$

$$\Rightarrow (x-y)+(y-y) = k+k'$$

$$\Rightarrow k+k' = 0$$

$\Rightarrow k = k' = 0,$ because k and $k' \in \mathbb{N}$ (0,5)

$$\Rightarrow x = y$$

So, S is antisymmetric.

S is transitive $\Leftrightarrow \forall x,y,z \in \mathbb{R}, (xSy \wedge ySz \Rightarrow xsz)$. (0,25)

Let $x,y,z \in \mathbb{R}$.

$$\begin{cases} xSy \\ ySz \end{cases} \Rightarrow \begin{cases} x-y \in \mathbb{N} \\ y-z \in \mathbb{N} \end{cases} \Rightarrow \begin{cases} \exists k \in \mathbb{N} : x-y = k \\ \exists k' \in \mathbb{N} : y-z = k' \end{cases}$$

$$\Rightarrow (x-y)+(y-z) = k+k'$$

$$\Rightarrow x-z = k+k'$$

$\Rightarrow \exists k'' = k+k' \in \mathbb{N} : x-z = k''$ (0,5)

$$\Rightarrow x-z \in \mathbb{N}$$

$$\Rightarrow xSz$$

So, S is transitive.

As S is reflexive, antisymmetric and transitive, then it is an order relation. (0,25)

(4) Is this order total?

This order is partial, because

$\exists x = \frac{1}{2},\ y = \frac{1}{3} \in \mathbb{R},$ such that $x - y = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \notin \mathbb{N}$ and $x - y = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6} \notin \mathbb{N}$.

That is, $\exists x = \frac{1}{2},\ y = \frac{1}{3} \in \mathbb{R},$ such that x and y are not comparable $\left(x \not{S} y\right.$ and $\left.y \not{S} x\right)$. (0,5)

Good luck