Continuous control (replacement)

Exercise 1 $(5\ pts)$

Let P, Q and R be propositions. Show that

$$\left[\left(P \wedge \overline{Q}\right) \Rightarrow R\right] \Leftrightarrow \left[\overline{P} \vee \left(\overline{Q} \Rightarrow R\right)\right]$$

using :

(1) The truth table.

(2) The definition of an implication, its negation, De Morgane's laws and the properties of logical connectors.

Exercise 2 $(4\ pts)$

Write in everyday language (English) the following mathematical formulas and say if they are true or false :

$$(A)\ \forall x \in \mathbb{R}^+, \exists n \in \mathbb{N} : n > x$$

$$(B)\ \exists! n \in \mathbb{N} : \forall k \in \mathbb{N},\ n \leq k.$$

Exercise 3 $(6\ pts)$

For $n \in \mathbb{N}$, we consider the proposition

$$A :\ n^3 \text{ is even} \Rightarrow\ n \text{ is even.}$$

(1) Write the negation of the proposition A.

(2) Write the contrapositive of the proposition A.

(3) Show that proposition A is true.

(4) Show by contradiction that $\sqrt[3]{2}$ is irrational number.

Exercise 4 $(5\ pts)$

We consider the two sets E and F defind by :

$$E = \left\{x \in \mathbb{R} : \left|1 - \frac{x}{2}\right| \leq 1\right\} \text{ and } F = \left\{x \in \mathbb{R} : \exists t \in \mathbb{R}^+,\ x = t + 2\right\}.$$

(1) Write E as an interval $[a, b]$.

(2) Show that $F = [2, +\infty[$.

(3) Determine $E \cap F$, $E \cup F$, $E \setminus F$, $F \setminus E$ et $C_{\mathbb{R}^+}(E)$, the complement of E relative to $\mathbb{R}^+$.

Contrôle continu (remplacement)

Exercice 1 $(5\ pts)$

Soient P, Q et R des propositions. Montrer que

$$[(P \wedge \overline{Q}) \Rightarrow R] \Leftrightarrow \left[\overline{P} \vee \left(\overline{Q} \Rightarrow R\right)\right]$$

en utilisant :

(1) La table de vérité.

(2) La définition d'une implication, sa négation, les lois de De Morgane et les propriétés des connecteurs logiques.

Exercice 2 $(4\ pts)$

Ecrire en langage courant (Français) les formules mathématiques suivantes et dire si elles sont vraies ou fausses :

$$(A)\ \forall x \in \mathbb{R}^+, \exists n \in \mathbb{N} : n > x$$

$$(B)\ \exists! n \in \mathbb{N} : \forall k \in \mathbb{N},\ n \leq k.$$

Exercice 3 $(6\ pts)$

Pour $n \in \mathbb{N}$, on considère la proposition

$$A :\ n^3 \text{ est pair} \Rightarrow\ n \text{ est pair.}$$

(1) Ecrire la négation de la proposition A.

(2) Ecrire la contraposée de la proposition A.

(3) Montrer que la proposition A est vraie.

(4) Montrer par l'absurde que $\sqrt[3]{2}$ est un nombre irrationnel.

Exercice 4 $(5\ pts)$

On considère les deux ensembles E et F définis par :

$$E = \left\{x \in \mathbb{R} : \left|1 - \frac{x}{2}\right| \leq 1\right\} \text{ et } F = \left\{x \in \mathbb{R} : \exists t \in \mathbb{R}^+,\ x = t + 2\right\}.$$

.

(1) Ecrire E sous la forme d'un intervalle $[a,b]$.

(2) Montrer que $F = [2, +\infty[$.

(3) Déterminer $E \cap F$, $E \cup F$, $E \setminus F$, $F \setminus E$ et $C_{\mathbb{R}^+}(E)$, le complémentaire de E par rapport à $\mathbb{R}^+$.

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Faculty of Sciences-Department of Mathematics — Module: Algebra 1 L1 Math+MI

Standard answer of continuous control

Exercise 1 $(5\ pts)$

Let P, Q and R be propositions. Let' Show that :

$$\left[(P \wedge \overline{Q}) \Rightarrow R\right] \Leftrightarrow \left[\overline{P} \vee (\overline{Q} \Rightarrow R)\right]$$

(1) Using the truth table :

P	Q	R	$\overline{P}$	$\overline{Q}$	$P \wedge \overline{Q}$	$\overline{Q} \Rightarrow R$	$(P \wedge \overline{Q}) \Rightarrow R$	$\overline{P} \vee (\overline{Q} \Rightarrow R)$
1	1	1	0	0	0	1	1	1
1	1	0	0	0	0	1	1	1
1	0	1	0	1	1	1	1	1
0	1	1	1	0	0	1	1	1
1	0	0	0	1	1	0	0	0
0	1	0	1	0	0	1	1	1
0	0	1	1	1	0	1	1	1
0	0	0	1	1	0	0	1	1

(2)

We notice that the propositions $\left[(P \wedge \overline{Q}) \Rightarrow R\right]$ and $\left[\overline{P} \vee (\overline{Q} \Rightarrow R)\right]$ have the same truth table, then they are eqivalent, that is $\left[(P \wedge \overline{Q}) \Rightarrow R\right] \Leftrightarrow \left[\overline{P} \vee (\overline{Q} \Rightarrow R)\right]$ (0,5)

(2) Using the definition of an implication, its negation, De Morgane's laws and the properties of logical connectors.

$\left[(P \wedge \overline{Q}) \Rightarrow R\right] \Leftrightarrow \overline{(P \wedge \overline{Q})} \vee R$ $\left(\text{definition of an implication : } (A \Rightarrow B) \Leftrightarrow \overline{A} \vee B\right)$ (0,5)

$\Leftrightarrow (\overline{P} \vee Q) \vee R$ $\left(\text{De Morgane's laws : } \overline{(A \wedge B)} \Leftrightarrow (\overline{A} \vee \overline{B}) \text{ and } \overline{(\overline{B})} \Leftrightarrow B\right)$ (0,5)

$\Leftrightarrow \overline{P} \vee (Q \vee R)$ $\left(\vee \text{ is associative, that is : } (A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)\right)$ (0,5)

$\Leftrightarrow \overline{P} \vee \left(\overline{(\overline{Q})} \vee R\right)$ $\left(Q \Leftrightarrow \overline{(\overline{Q})}\right)$ (0,5)

$\left[(P \wedge \overline{Q}) \Rightarrow R\right] \Leftrightarrow \overline{P} \vee (\overline{Q} \Rightarrow R)$ $\left(\text{definition of an implication : } (\overline{A} \vee B) \Leftrightarrow (A \Rightarrow B)\right)$ (0,5)

Exercise 2 $(4\ pts)$

We write in everyday language (English) the following mathematical formulas and say if they are true or false :

$$(A)\ \forall x \in \mathbb{R}^{+}, \exists n \in \mathbb{N} : n > x$$

$$(B)\ \exists! n \in \mathbb{N} : \forall k \in \mathbb{N},\ n \leq k.$$

(A) For every positive real number x, there exists a natural number n such that n is greater than x. (1)

Pour tout nombre réel positif x, il existe un nombre naturel n tel que n est supérieur à x.

The proposition (A) is true, because $\forall x \in \mathbb{R}^{+}$, $\exists n = E(x) + 1 \in \mathbb{N}$, such that $n = E(x) + 1 > x$ (1)
where $E(x)$, is the integer part of x. $\left(\text{we have } E(x) \leq x < E(x) + 1\right)$

(B) There exists an unique natural number n, such that for every natural number k, we have n is less than or equal to k. (1)

Il existe un unique nombre naturel n, tel que pour tout nombre naturel k, on a n inférieur ou égal à k. (1)

The proposition (B) is true, because $\exists! n = 0 \in \mathbb{N}$, such that for all $k \in \mathbb{N}$, we have $0 \leq k$.

Exercise 3 $(6\ pts)$

For $n \in \mathbb{N}$, we consider the proposition

$$A: \ n^3 \text{ is even} \Rightarrow \ n \text{ is even.}$$

(1) Write the negation of the proposition A.

$\overline{A}: \overline{n^3 \text{ is even} \Rightarrow \ n \text{ is even}} \Leftrightarrow \left(\overline{n^3 \text{ is even}}\right) or \left(n \text{ is even}\right) \Leftrightarrow \left(n^3 \text{ is odd}\right) or \left(n \text{ is even}\right)$ (1)

(2) Write the contrapositive of the proposition $A.((P \Rightarrow Q) \Leftrightarrow (\overline{Q} \Rightarrow \overline{P}))$

$\left[\left(n^3 \text{ is even}\right) \Rightarrow \left(n \text{ is even}\right)\right] \Leftrightarrow \overline{\left(n \text{ is even}\right)} \Rightarrow \overline{\left(n^3 \text{ is even}\right)} \Leftrightarrow \left(n \text{ is odd}\right) \Rightarrow \left(n^3 \text{ is odd}\right)$ (0,5)

(3) Show that proposition A is true.

Let' show that $\left(n \text{ is odd}\right) \Rightarrow \left(n^3 \text{ is odd}\right)$

n is odd $\Rightarrow \exists k \in \mathbb{N},\ n = 2k+1.$ (0,5)

$n^3 = (2k+1)^3 = (2k)^3 + 3(2k)^2.1 + 3(2k).1^2 + 1^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1.$ (1)

then, $\exists l = (4k^3 + 6k^2 + 3k) \in \mathbb{N} : n^3 = 2l + 1,$ which proves that n^3 is odd. (0,5)

Therefore, $\left(n^3 \text{ is even}\right) \Rightarrow \left(n \text{ is even}\right).$ (0,5)

(4) Show by contradiction that $\sqrt[3]{2}$ is irrational number.

We assume that : $\sqrt[3]{2}$ is rational number.

$\sqrt[3]{2} \in \mathbb{Q} \Rightarrow \sqrt[3]{2} = \frac{a}{b}$ with a and b are natural numbers that are prime to each other. (0,5)

$\Rightarrow 2 = \frac{a^3}{b^3} \Rightarrow a^3 = 2b^3 \Rightarrow a^3$ is even $\Rightarrow a$ is even. (0,5)

So $a = 2k$ with $k \in \mathbb{N}$

Hence, $a^3 = 2b^3 \Rightarrow (2k)^3 = 2b^3 \Rightarrow 8k^3 = 2b^3 \Rightarrow b^3 = 4k^3$
$\Rightarrow b^3$ is even $\Rightarrow b$ is even. (0,5)

a and b are both even, which contradicts the hypothesis $\left(a \text{ and } b \text{ are prime to each other}\right).$ (0,5)

Conclusion : $\sqrt[3]{2}$ is irrational. (0,5)

Exercise 4 $(5\ pts)$

We consider the two sets E and F defind by :

$$E = \left\{x \in \mathbb{R} : \left|1 - \frac{x}{2}\right| \leq 1\right\} \text{ and } F = \left\{x \in \mathbb{R} : \exists t \in \mathbb{R}^+,\ x = t + 2\right\}$$

.

(1) Write E as an interval $[a,b]$.

$x \in E \Leftrightarrow \left|1 - \frac{x}{2}\right| \leq 1 \Leftrightarrow -1 \leq 1 - \frac{x}{2} \leq 1 \quad \Leftrightarrow -1 - 1 \leq 1 - \frac{x}{2} - 1 \leq 1 - 1$ (0,5)

$\Leftrightarrow -2 \leq -\frac{x}{2} \leq 0 \Leftrightarrow 0 \leq \frac{x}{2} \leq 2 \Leftrightarrow 0 \leq x \leq 4$ (0,5)

Then $E = [0,4].$

(2) Show that $F = [2,+\infty[.$

$(i)\ F \subset [2,+\infty[.$

$x \in F \Rightarrow \exists t \in \mathbb{R}^+,\ x = t + 2 \Rightarrow x \geq 2 \Rightarrow x \in [2,+\infty[$

then $F \subset [2,+\infty[.$ (0,5)

$(ii)\ [2,+\infty[\subset F.$

$x \in [2,+\infty[\Rightarrow x \geq 2 \Rightarrow x - 2 \geq 0 \Rightarrow \exists t \in \mathbb{R}^+, x - 2 = t \Rightarrow \exists t \in \mathbb{R}^+, x = t + 2 \Rightarrow x \in F$

then $[2,+\infty[\subset F.$ (0,5)

As $F \subset [2,+\infty[$ and $[2,+\infty[\subset F,$ then $F = [2,+\infty[.$

(3) Determine $E \cap F, E \cup F, E \setminus F, F \setminus E$ et $C_{\mathbb{R}^+}(E)$. lthe complement of E relative to $\mathbb{R}^+$.

$E \cap F = [0,4] \cap [2,+\infty[= [2,4].$ (0,5)

$E \cup F = [0,4] \cup [2,+\infty[= [0,+\infty[.$ (0,5)

$E \setminus F = [0,4] \setminus [2,+\infty[= [0,2[.$ (0,5)

$F \setminus E = [2,+\infty[\setminus [0,4] =]4,+\infty[.$ (0,5)

$C_{\mathbb{R}^+}(E) = \mathbb{R}^+ \setminus E = \mathbb{R}^+ \setminus [0,4] =]4,+\infty[.$ (0,5)