Université Abou Bekr Belkaid de Tlemcen — S1 LMD **M et MI** (2023/2024)

Faculté des Sciences — 1 heure - le 14 décember 2023.

Contrôle Continu d'Analyse 1

Question de Cours (4 points)

Trouver le module et l'argument du nombre complexe : $Z = \left(e^{i\theta} + e^{2i\theta}\right)$, $\theta \in]\pi, 2\pi]$.

Exercice 1 (6 points)

Soit l'ensemble $E = \left\{ \frac{\cos(m\pi)}{m} + e^{-|n|}, n \in \mathbb{Z}, m \in \mathbb{Z}^* \right\}$.

1. Montrer que E est borné. **(3 points)**
2. Déterminer $\sup E$, $\inf E$ et $\max E$, $\min E$ s'ils existent. **(3 points)**

Exercice 2 (5 points)

Trouver les limites des suites suivantes définies par leurs termes généraux :

$$1/U_n = \cos(n) \sin\left(\frac{1}{n}\right), (n \in \mathbb{N}^*) . \textbf{(2 points)} \quad ; \quad 2/V_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}, (n \in \mathbb{N}^*) . \textbf{(3 points)}$$

Le téléphone portable est strictement interdit. Bon courage.

Abou Bekr Belkaid University of Tlemcen. — 1st grade LMD **M and MI** (2023/2024)

Faculty of Sciences — 1 hour - the 14^{th} of december 2023.

Continuous Control of Calculus 1

Theoritical Question (4 points)
Find the modulus and the argument of the complex number : $Z = \left(e^{i\theta} + e^{2i\theta}\right), \theta \in]\pi, 2\pi]$.

Exercise 1 (6 points)

Let be the set $E = \left\{ \frac{\cos(m\pi)}{m} + e^{-|n|}, n \in \mathbb{Z}, m \in \mathbb{Z}^* \right\}$.

1. Show that E is bounded. **(3 points)**
2. Find $\sup E$, $\inf E$ and $\max E, \min E$ if they exist. **(3 points)**

Exercise 2 (5 points)

Find the limits of the following sequences defined by them general terms:

$$1/U_n = \cos(n) \sin\left(\frac{1}{n}\right), (n \in \mathbb{N}^*) . \textbf{ (2 points)} \; ; \quad 2/V_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}, (n \in \mathbb{N}^*) . \textbf{(3 points)}$$

Cell phones are strictly prohibited. Good luck.

Correction of the Continuous Control of Calculus 1

Question de Cours (4 points)

Find the modulus and the argument of the complex number : $Z = \left(e^{i\theta} + e^{2i\theta}\right), \theta \in]\pi, 2\pi]$.

$$\begin{aligned} Z &= \left(e^{i\theta} + e^{2i\theta}\right) \\ &= e^{\frac{3i\theta}{2}}\left(e^{\frac{-\theta}{2}i} + e^{\frac{\theta}{2}i}\right) (\mathbf{1\ point}) \\ &= 2\cos\left(\frac{\theta}{2}\right) e^{\frac{3i\theta}{2}} (\mathbf{1\ point}) \end{aligned}$$

Because $\forall z \in \mathbb{C}, z = e^{i\alpha} \Rightarrow \overline{z} = e^{-i\alpha}, (\alpha \in \mathbb{R})$ and $\forall z \in \mathbb{C}, z + \overline{z} = 2\,\mathrm{Re}(z)$.

Considering that $z = \cos(\alpha) + i\sin(\alpha), (i.e. |z| = 1)$, then $z + \overline{z} = 2\cos(\alpha)$.

Since $\theta \in]\pi, 2\pi]$, then $\dfrac{\theta}{2} \in \left]\frac{\pi}{2}, \pi\right]$ which means that $\cos\left(\frac{\theta}{2}\right) < 0$. So,

$$\begin{aligned} Z &= -2\left(-\cos\left(\frac{\theta}{2}\right)\right) e^{\frac{3i\theta}{2}} \\ &= e^{i\pi}2\left(-\cos\left(\frac{\theta}{2}\right)\right) e^{\frac{3i\theta}{2}} \\ &= -2\cos\left(\frac{\theta}{2}\right) e^{i\left(\frac{3\theta}{2}+\pi\right)}. (\mathbf{1\ point}) \end{aligned}$$

In conclusion

$$\begin{cases} |Z| = -2\cos\left(\frac{\theta}{2}\right) = \left|2\cos\left(\frac{\theta}{2}\right)\right| \\ \arg(Z) = \left(\frac{3\theta}{2} + \pi\right) \end{cases}, (\theta \in]\pi, 2\pi]). (\mathbf{1\ point})$$

Exercise 1 (6 points)

Let be the set $E = \left\{\dfrac{\cos(m\pi)}{m} + e^{-|n|}, n \in \mathbb{Z}, m \in \mathbb{Z}^*\right\}$.

1. Show that E is bounded.

   Method 1 : One has

$$\begin{aligned} \left|\frac{\cos(m\pi)}{m} + e^{-|n|}\right| &\leq \left|\frac{\cos(m\pi)}{m}\right| + \left|e^{-|n|}\right| \\ &\leq \frac{1}{|m|} + e^{-|n|} \\ &\leq 2 \end{aligned}$$

   Because

$$\begin{cases} |\cos(m\pi)| \leq 1, \forall m \in \mathbb{Z}^* \\ e^{-|n|} \leq 1, \forall n \in \mathbb{Z}. \\ \frac{1}{|m|} \leq 1, \forall m \in \mathbb{Z}^* \end{cases}.$$

   So, $\exists \alpha = 2 > 0 / \ \forall n \in \mathbb{Z}, \forall m \in \mathbb{Z}^*, \left|\dfrac{\cos(m\pi)}{m} + e^{-|n|}\right| \leq 2$. Which means that E is bounded by 2.(**3 points**)

   Method 2: One has

$$\begin{cases} -1 \leq \cos(m\pi) \leq 1, \forall m \in \mathbb{Z}^* \\ -1 \leq \frac{1}{m} \leq 1, \forall m \in \mathbb{Z}^* \end{cases}$$

$$\Rightarrow -1 \leq \frac{\cos(m\pi)}{m} \leq 1.$$

(One may study positive then negative cases).

And since

$$\forall n \in \mathbb{Z}, 0 < e^{-|n|} \leq 1$$

then,

$$-1 \leq \frac{\cos(m\pi)}{m} + e^{-|n|} \leq 2.$$

So, $\exists \alpha = -1$ and $\beta = 2 / \ \forall n \in \mathbb{Z}, \forall m \in \mathbb{Z}^*, -1 < \dfrac{\cos(m\pi)}{m} + e^{-|n|} \leq 2.$ (**3 points**).

2. Find $\sup E$, $\inf E$ and $\max E, \min E$ if they exist.

We may write E as $E = E_1 + E_2$ where : $E_1 = \left\{\dfrac{\cos(m\pi)}{m}, m \in \mathbb{Z}^*\right\}$ and $E_2 = \left\{e^{-|n|}, n \in \mathbb{Z}\right\}$.

One has: $\forall m \in \mathbb{Z}^*$, $-1 \leq \dfrac{\cos(m\pi)}{m} \leq 1$.

and when $m = -1$, $\dfrac{\cos(m\pi)}{m} = 1$, so $1 \in E_1$ which means that : $\max E_1 = 1 \Rightarrow \sup E_1 = 1$(**0.25 point**)

and when $m = 1$, $\dfrac{\cos(m\pi)}{m} = -1$, so $(-1) \in E_1$, which means that : $\min E_1 = -1 \Rightarrow \inf E_1 = -1$.(**0.25 point**)

Now, since $\forall n \in \mathbb{Z}, 0 < e^{-|n|} \leq 1$ and $n = 0 \Rightarrow e^{-|n|} = 1$, so $1 \in E_2$,
which means that : $\max E_2 = 1 \Rightarrow \sup E_2 = 1$. (**0.25 point**)

But $0 \notin E_2 (i.e. \nexists n \in \mathbb{Z}, / 0 = e^{-|n|})$, so we have to show that : $\inf E_2 = 0$:

Using the infimum characterization :

$$\forall \varepsilon > 0, \exists x \in E_2 / 0 \leq x < 0 + \varepsilon \quad (\textbf{0.25 point})$$

Let be $\varepsilon > 0, \exists ? p \in \mathbb{N} / \ e^{-p} < \varepsilon$

$$\begin{aligned} e^{-p} &< \varepsilon \\ &\Rightarrow -p < \ln \varepsilon \\ &\Rightarrow p > -\ln \varepsilon \\ &\Rightarrow p > \ln \frac{1}{\varepsilon} \end{aligned}$$

p exists because $\mathbb{R}$ is Archimedean. It is sufficient to take : $p = E\left(\ln \frac{1}{\varepsilon}\right) + 1$.(**0.5 point**)

So:

$$\forall \varepsilon > 0, \exists p = E\left(\ln \frac{1}{\varepsilon}\right) + 1 \quad \in \mathbb{N} \quad / \quad e^{-p} < \varepsilon$$

In concucion $\inf E_2 = 0$.(**0.25 point**)

Furthermore, since $0 \notin E_2$ then $\min E_2$ doesn't exist.(**0.5 point**)

Finelly :

$$\begin{aligned} \sup E &= \sup(E_1 + E_2) = \sup E_1 + \sup E_2 = 2 = \max E (\textbf{0.25 point}) \\ \inf E &= \inf(E_1 + E_2) = \inf E_1 + \inf E_2 = -1 (\textbf{0.25 point}) \\ &\min E \text{ doesn't exist.} (\textbf{0.25 point}) \end{aligned}$$

Exercise 2 (**5 points**)

Find the limit of the following sequences defined by them general terms:

$$1/U_n = \cos(n)\sin\left(\frac{1}{n}\right), (n \in \mathbb{N}^*). \quad 2/V_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}, (n \in \mathbb{N}^*).$$

1/ Since $\forall n \in \mathbb{N}^*, |\cos(n)| \leq 1$ and $\sin\left(\frac{1}{n}\right) \longrightarrow 0$ then $\lim\left[\cos(n)\sin\left(\frac{1}{n}\right)\right] = 0$.(**2 points**)

2/ One has :

$$\begin{aligned}
1 &\leq k \leq n \\
&\Rightarrow 1+n^2 \leq k+n^2 \leq n+n^2 \\
&\Rightarrow \sqrt{1+n^2} \leq \sqrt{k+n^2} \leq \sqrt{n+n^2} \\
&\Rightarrow \frac{1}{\sqrt{n+n^2}} \leq \frac{1}{\sqrt{k+n^2}} \leq \frac{1}{\sqrt{1+n^2}} \\
&\Rightarrow \sum_{k=1}^{n}\frac{1}{\sqrt{n+n^2}} \leq \sum_{k=1}^{n}\frac{1}{\sqrt{k+n^2}} \leq \sum_{k=1}^{n}\frac{1}{\sqrt{1+n^2}} \\
&\Rightarrow \left(\sum_{k=1}^{n}1\right)\left(\frac{1}{\sqrt{n+n^2}}\right) \leq V_n \leq \left(\sum_{k=1}^{n}1\right)\left(\frac{1}{\sqrt{1+n^2}}\right)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left(\frac{n}{\sqrt{n+n^2}}\right) \leq V_n \leq \left(\frac{n}{\sqrt{1+n^2}}\right) \\
&\Rightarrow \lim\left(\frac{n}{\sqrt{n+n^2}}\right) \leq \lim V_n \leq \lim\left(\frac{n}{\sqrt{1+n^2}}\right) \\
&\Rightarrow \lim\left(\frac{n}{\sqrt{n^2}}\right) \leq \lim V_n \leq \lim\left(\frac{n}{\sqrt{n^2}}\right) \\
&\Rightarrow 1 \leq \lim V_n \leq 1 \\
&\Rightarrow \lim V_n = 1 \text{ according to the three sequences theorem.}(\textbf{3 points})
\end{aligned}$$