Continuous control

Exercise 1 (5 pts)

(1) Let P, Q and R be propositions. Complete the following truth table :

P	Q	R	$P \vee Q$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \vee Q) \Rightarrow R$	$(P \Rightarrow R) \vee (Q \Rightarrow R)$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$

(2) Using this table, which of the following three propositions are equivalent?
$[(P \vee Q) \Rightarrow R]$, $[(P \Rightarrow R) \vee (Q \Rightarrow R)]$ and $[(P \Rightarrow R) \wedge (Q \Rightarrow R)]$.

(3) Give each of the following mathematical propositions in formula, then its negation in formula.

P : There exists a non-zero natural number that divides every non-zero relative integer.

Q : Any part of the set of natural numbers contains the natural number zero.

R : The equation $x^2 - 2 = 0$ admits at least one rational solution.

Exercise 2 (5 pts)

(1) Write the following set in extension : $A = \{x \in \mathbb{R} : x^2 + x + 1 = 0\}$.

(2) Write the following set in comprehension : $B = \{-1, 1\}$.

(3) Let $E = \{1,2,3\} \subset \mathbb{N}$. Find $P(E)$ the set of power of E.

(4) Let A, B and C be parts of $P(E)$ given by :

$A = \{X \in P(E) : 1 \in X\}$, $B = \{X \in P(E) : 2 \notin X\}$, $C = \{X \in P(E) : 1 \notin X$ and $3 \in X\}$.

- Give the sets A, B and C in extension.

(5) Find

$A \cap B$, $.A \setminus B$, $B \cup C$, and $\overline{C} = C_{P(E)}(C)$, the complement of C relative to $P(E)$.

Exercise 3 (5 pts)

For $n \in \mathbb{N}^*$. Consider the statement : If $n^2 - 1$ is not a multiple of 4, then n is even.

(1) Write this statement in the form of an implication $P \Rightarrow Q$.

(2) Write the contrapositive and converse of this implication $P \Rightarrow Q$.

(3) Show the implication $P \Rightarrow Q$ by contraposition.

(4) Is the converse of the implication $P \Rightarrow Q$ true?

Exercise 4 (5 pts)

For $n \in \mathbb{N}^* \setminus \{1\}$, we pose $S_n = \sum_{k=1}^{n-1} 2^k = 2 + 2^2 + 2^3 + \ldots + 2^{n-1}$.

(1) By calculating $2S_n - S_n$, show that : $S_n = 2^n - 2$.

(2) Show by induction that : $\forall n \in \mathbb{N}^* \setminus \{1\}$, $S_n = \sum_{k=1}^{n-1} 2^k = 2^n - 2$.

Good luck

Université Abou Bekr Belkaid - Tlemcen — Année universitaire: 2023 - 2024

Faculté des sciences-département de mathématiques — Module: Algèbre 1 L1 Math+MI

Contrôle continu

Exercice 1 $(5\ pts)$

(1) Soient P,Q et R des propositions. Compléter la table de vérité suivante :

P	Q	R	$P \vee Q$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \vee Q) \Rightarrow R$	$(P \Rightarrow R) \vee (Q \Rightarrow R)$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$

(2) A l'aide de cette table, quelles propositions sont équivalentes parmi les trois propositions suivantes :
$[(P \vee Q) \Rightarrow R]$, $[(P \Rightarrow R) \vee (Q \Rightarrow R)]$ et $[(P \Rightarrow R) \wedge (Q \Rightarrow R)]$.

(3) Donner chacune des propositions mathématiques suivantes en formule, puis sa négationn en formule.

P : Il existe un entier naturel non nul qui divise chaque entier relatif non nul.

Q : Toute partie de l'ensemble des entiers naturels contient l'entier naturel zéro.

R : L'équation $x^2 - 2 = 0$ admet au moins une solution rationnelle.

Exercice 2 $(5\ pts)$

(1) Ecrire en extension l'ensemble suivant : $A = \{x \in \mathbb{R} : x^2 + x + 1 = 0\}$

(2) Ecrire en compréhension l'ensemble suivant : $B = \{-1, 1\}$.

(3) Soit $E = \{1,2,3\} \subset \mathbb{N}$. Déterminer $P(E)$ l'ensemble des parties de E.

(4) Soient A, B et C des parties de $P(E)$ données par :
$A = \{X \in P(E) : 1 \in X\}$, $B = \{X \in P(E) : 2 \notin X\}$, $C = \{X \in P(E) : 1 \notin X$ et $3 \in X\}$.

- Donner les ensembles A, B et C en extension.

(5) Déterminer
$A \cap B$, $A \setminus B$, $B \cup C$ et $\overline{C} = C_{P(E)}(C)$, le complémentaire de C par rapport à $P(E)$.

Exercice 3 $(5\ pts)$

Pour $n \in \mathbb{N}^*$. Considérons l'énoncé : Si $n^2 - 1$ n'est pas multiple de 4, alors n est pair.

(1) Ecrire cet énoncé sous la forme d'une implication $P \Rightarrow Q$.

(2) Ecrire la contraposée et la réciproque de cette implication $P \Rightarrow Q$.

(3) Montrer l'implication $P \Rightarrow Q$ par contraposition.

(4) La réciproque de l'implicatrion $P \Rightarrow Q$ est-elle vraie?

Exercice 4 $(5\ pts)$

Pour $n \in \mathbb{N}^* \setminus \{1\}$, on pose $S_n = \sum_{k=1}^{n-1} 2^k = 2 + 2^2 + 2^3 + \ldots + 2^{n-1}$.

(1) En calculant $2S_n - S_n$, montrer que : $S_n = 2^n - 2$.

(2) Montrer par récurrence que : $\forall n \in \mathbb{N}^* \setminus \{1\}$, $S_n = \sum_{k=1}^{n-1} 2^k = 2^n - 2$.

Bon courage

Abou Bekr Belkaid University - Tlemcen — Academic year: 2023 - 2024
Faculty of Sciences-Department of Mathematics — Module: Algebra 1 L1
Math+MI

Continuous control

Exercise 1 (5)

(1) Let P, Q and R be propositions. Complete the following truth table :

P	Q	R	$P \vee Q$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \vee Q) \Rightarrow R$	$(P \Rightarrow R) \vee (Q \Rightarrow R)$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
1	1	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	0
1	0	1	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	1	0
0	1	0	1	1	0	0	1	0
0	0	1	0	1	1	1	1	1
0	0	0	0	1	1	1	1	1

(2)

(2) Using this table, which of the following three propositions are equivalent?
$[(P \vee Q) \Rightarrow R], [(P \Rightarrow R) \vee (Q \Rightarrow R)]$ and $[(P \Rightarrow R) \wedge (Q \Rightarrow R)]$.

$$[(P \vee Q) \Rightarrow R] \Leftrightarrow [(P \Rightarrow R) \wedge (Q \Rightarrow R)]$$ (0,5)

.

(3) Give each of the following mathematical propositions in formula, then its negation in formula.
P : There exists a non-zero natural number that divides every non-zero relative integer.

$P : \exists n \in \mathbb{N}^*, \forall k \in \mathbb{Z}^* : n \mid k.$
$\overline{P} : \forall n \in \mathbb{N}^*, \exists k \in \mathbb{Z}^* : n$ does not divide k. (1)

Q : Any part of the set of natural numbers contains the natural number zero.

$Q : \forall A \subset \mathbb{N}, 0 \in A.$
$\overline{Q} : \exists A \subset \mathbb{N}, 0 \notin A.$ (0,5+0,25)

R : The equation $x^2 - 2 = 0$ admits at least one rational solution.

$R : \exists x \in \mathbb{Q}, x^2 - 2 = 0.$
$\overline{R} : \forall x \in \mathbb{Q}, x^2 - 2 \neq 0.$ (0,5+0,25)

Exercise 2 (5)

(1) Write the following set in extension : $A = \{x \in \mathbb{R} : x^2 + x + 1 = 0\}$.

The equation $x^2 + x + 1 = 0$ does not admit real solutions, because its discriminant $\triangle < 0$. Hence, $A = \varnothing$. (0,5)

(2) Write the following set in comprehension : $B = \{-1, 1\}$.

$$B = \{(-1)^n,\ n \in \mathbb{N}\} = \{x \in \mathbb{R},\ |x| = 1\}.$$ (0,5)

(3) Let $E = \{1,2,3\} \subset \mathbb{N}$. Find $P(E)$ the set power of E.

$$P(E) = \{\varnothing,\ \{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}.$$ (0,5)

(4) Let A, B and C be parts of $P(E)$ given by :

$A = \{X \in P(E) :\ 1 \in X\}$, $B = \{X \in P(E) :\ 2 \notin X\}$, $C = \{X \in P(E) :\ 1 \notin X$ and $3 \in X\}$.

- Give the sets A,B and C in extension.

$$A = \{\{1\},\{1,2\},\{1,3\},\{1,2,3\}\}.$$
$$B = \{\varnothing,\ \{1\},\{3\},\{1,3\}\}.$$
$$C = \{\{3\},\{2,3\}\}.$$
(1,5)

(5) Find $A \cap B, A \setminus B, B \cup C, \overline{C} = C_{P(E)}(C)$ the complement of C relative to $P(E)$.

$$A \cap B = \{\{1\},\{1,3\}\}.$$
$$A \setminus B = \{\{1,2\},\{1,2,3\}\}.$$
$$B \cup C = \{\varnothing,\ \{1\},\{3\},\{1,3\},\{2,3\}\}.$$
$$\overline{C} = C_{P(E)}(C) = \{\varnothing,\ \{1\},\{2\},\{1,2\},\{1,3\},\{1,2,3\}\}.$$
(2)

Exercise 3 (5)

For $n \in \mathbb{N}^*$. Consider the statement : If $n^2 - 1$ is not a multiple of 4, then n is even.

(1) Write this statement in the form of an implication $P \Rightarrow Q$.

$P \Rightarrow Q :$ ($n^2 - 1$ is not a multiple of 4) $\Rightarrow$ (n is even). (0,5)

(2) Write the contrapositive and converse of this implication $P \Rightarrow Q$.

The contrapositive : (n is odd) $\Rightarrow$ ($n^2 - 1$ is a multiple of 4).

The converse : (n is even) $\Rightarrow$ ($n^2 - 1$ is not a multiple of 4). (1)

(3) Show the implication $P \Rightarrow Q$ by contraposition.

We know that $(P \Rightarrow Q) \Leftrightarrow (\overline{Q} \Rightarrow \overline{P})$.
We assume that n is odd.
n is odd $\Rightarrow \exists k \in \mathbb{N} : n = 2k + 1$.
We have, $n^2 - 1 = (2k+1)^2 - 1 = (4k^2 + 4k + 1) - 1 = 4k^2 + 4k$ (2)
$= 4(k^2 + k) = 4l$, with $l = k^2 + k \in \mathbb{N}$.
So, $n^2 - 1$ is a multiple of 4. Therfore $\left(n \text{ is even}\right) \Rightarrow (n^2 - 1 \text{ is a multiple of } 4)$.
Consequnetly, $(n^2 - 1 \text{ is not a multiple of } 4) \Rightarrow \left(n \text{ is even}\right)$.
(4) Is the converse of the implication $P \Rightarrow Q$ true?

Assume that n is even.
n is even $\Rightarrow \exists k \in \mathbb{N} : n = 2k$.
We have, $n^2 - 1 = (2k)^2 - 1 = (4k^2) - 1 = 4k^2 - 1$. (1,5)
So, $n^2 - 1$ is not a multiple of 4. Therfore $\left(n \text{ is even}\right) \Rightarrow (n^2 - 1 \text{ is not a multiple of } 4)$.
Consequnetly, $\left(n \text{ is even}\right) \Rightarrow (n^2 - 1 \text{ is not a multiple of } 4)$.
That is, the converse of the implication $P \Rightarrow Q$ is true.

Exercise 4 (5)

For $n \in \mathbb{N}^* \setminus \{1\}$, we pose $S_n = \sum_{k=1}^{n-1} 2^k = 2 + 2^2 + 2^3 + \ldots + 2^{n-2} + 2^{n-1}$.
(1) By calculating $2S_n - S_n$, show that : $S_n = 2^n - 2$.

$2S_n - S_n = 2(2 + 2^2 + 2^3 + \ldots + 2^{n-2} + 2^{n-1}) - (2 + 2^2 + 2^3 + \ldots + 2^{n-2} + 2^{n-1})$ (1,5)
$= (2^2 + 2^3 + 2^4 + \ldots + 2^{n-1} + 2^n) - (2 + 2^2 + 2^3 + \ldots + 2^{n-2} + 2^{n-1})$
$2S_n - S_n = 2^n - 2$
Hence $S_n = 2^n - 2$

(2) Show by induction that : $\forall n \in \mathbb{N}^* \setminus \{1\},\ S_n = \sum_{k=1}^{n-1} 2^k = 2^n - 2$.

1^{st} step (initialization).For $n = 2$, we have, $S_2 = \sum_{k=1}^{1} 2^k = 2^1 = 2$, and $2^2 - 2 = 4 - 2 = 2$ (0,5)

That is $S_2 = \sum_{k=1}^{1} 2^k = 2^1 - 2 = 2$.
Therfore, the property is true for $n = 2$.
2^{nd} step (heredity).We assume that : $S_n = 2^n - 2$, and let's show that : $S_{n+1} = 2^{n+1} - 2$. (0,5)
We have $S_{n+1} = \sum_{k=1}^{n} 2^k = 2 + 2^2 + 2^3 + \ldots + 2^{n-2} + 2^{n-1} + 2^n$ (0,5)
$= (2 + 2^2 + 2^3 + \ldots + 2^{n-2} + 2^{n-1}) + 2^n$
$= S_n + 2^n$ (0,5)
$= (2^n - 2) + 2^n$ (induction hypothesis)
$= 2.2^n - 2$ (1)
$S_{n+1} = 2^{n+1} - 2$
Hence, the property is true for $n + 1$
3^{rd} step (conclusion) By the principle of induction, we deduce that :
$\forall n \in \mathbb{N}^* \setminus \{1\},\ S_n = \sum_{k=1}^{n-1} 2^k = 2^n - 2$. (0,5)

Good luck