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Worksheet N°3
Number Sequences
LMD 1^{st} year 2023-2024

Exercise 1

1. Let be $(U_n)_{n\in\mathbb{N}^*}$ a number sequence, such that $U_n = \frac{n-2}{2n}$.
   a) By two different methods, sow that $(U_n)_{n\in\mathbb{N}^*}$ is bounded.
   b) $(U_n)_{n\in\mathbb{N}^*}$ is that monotone?

2. Give the mathematical definition of a sequence which is not increasing.

3. Show that, the general term sequence $V_n = n \times (-1)^n, n \in \mathbb{N}$,
   is not increasing. Is a non-increasing sequence decreasing?

Exercise 2

1. Using the definition of limit, show that

$$\lim_{n\to+\infty} \frac{(-1)^n}{n} = 0, \qquad \lim_{n\to+\infty} \frac{5n-1}{2n+1} = \frac{5}{2}, \qquad \lim_{n\to+\infty} \sqrt[n]{5} = 1, \qquad \lim_{n\to+\infty} \left(\frac{3}{2}\right)^n = +\infty.$$

2. Find the limit of the following sequences when it exists

$$\lim_{n\to+\infty} (1+\frac{1}{2}+\frac{1}{2^2}+....+\frac{1}{2^n}), \qquad \lim_{n\to+\infty} (n-\sqrt{(n+1)(n-2)}), \qquad \lim_{n\to+\infty} (\frac{3+(\pi\times 7^n)}{5-(e\times 7^n)}).$$

$$\lim_{n\to+\infty} (\sqrt[3]{n+1}-\sqrt[3]{n}), \quad \lim_{n\to+\infty} \left(\sum_{k=1}^{n} \frac{1}{k^2+k}\right), \quad \lim_{n\to+\infty} \frac{n^2\sin(n^3+1)}{n^3+1}, \quad \lim_{n\to+\infty} \left(\sum_{k=1}^{n} \frac{n}{n^2+k}\right)$$

Exercice 3

Let be $(U_n)_{n\in\mathbb{N}}$ a number sequence, such that $U_{n+1} = \dfrac{U_n}{2} + \dfrac{3}{2U_n}$ and $U_0 = 1$.

1. a) Show that, $\forall n \in \mathbb{N}$, $U_n \geq 0$,
   b) if it should converge what would be its limit?

2. let's note, $l = \lim\limits_{n\to+\infty} U_n$, Show that $\forall n \in \mathbb{N}$, $U_n - l > 0$.
   Deduce that (U_n) is decreasing, what to conclude?

Exercice 4
In each case, study whether the sequences are adjacent.
If so, determine their common limit if possible

1. $U_n = 3 - \dfrac{1}{n^2}$ and $V_n = 3 + \dfrac{1}{n^3}$, $\forall n \in \mathbb{N}^*$.
2. $U_n = 1 - \dfrac{1}{n}$ and $V_n = 1 + \sin(\dfrac{1}{n})$, $\forall n \in \mathbb{N}^*$.
3. $U_n = \sum\limits_{k=1}^{n} \dfrac{1}{k^2}$ and $V_n = U_n + \dfrac{1}{n}$, $\forall n \in \mathbb{N}^*$.

Exercise 5

1. Let be $(U_n)_{n\in\mathbb{N}}$ a number sequence, such that $U_{n+1} = \frac{4}{5}U_n + \frac{1}{5}$ and $U_0 = 0$.
   a) Show that $\forall n \in \mathbb{N}^*$, $|U_{n+1} - U_n| = (\frac{4}{5})^n\,|U_1 - U_0|$.
   b) Deduce that (U_n) is a Cauchy one, then calculate the limit.
2. Show that $U_n = \sum\limits_{k=2}^{n} \frac{1}{k^2}$, $n \in \mathbb{N}\backslash\{0,1\}$ is a Cauchy one, what can you deduce.
3. **(Additional)** Show that $U_n = \sum\limits_{k=1}^{n} \frac{\cos k}{k!}$ is a Cauchy one, what can you deduce.
4. Show that $U_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ is increasing and is not a Cauchy one, what can you deduce (Indication calculate $|U_{2n} - U_n|$).

Exercise 6 (**Additional**)

1. Let be $(U_n)_{n\in\mathbb{N}}$ a number sequence, such that
$$U_n = \frac{2n+1}{3n^2+1} + \frac{2n+1}{3n^2+2} + ... + \frac{2n+1}{3n^2+n}.$$
   Show that U_n is convergent and determine its limit.
2. Let be $(U_n)_{n\in\mathbb{N}}$ a number sequence, such that
$$U_n = \frac{1 \times 3 \times ... \times (2n+1)}{3 \times 6 \times ... \times (3n+3)}.$$
   Show that U_n is convergent and determine its limit..

Exercise 7 (**Additional**)
We consider the sequences $(U_n)_{n\in\mathbb{N}^*}$ and $(V_n)_{n\in\mathbb{N}^*}$ a numbers sequences, such that
$$U_n = 1 + \frac{1}{2^3} + \frac{1}{3^3} + ... + \frac{1}{n^3} \quad \text{and} \quad V_n = U_n + \frac{1}{n^2}.$$
Show that these two sequences are convergent and have the same limit (which we will not try to calculate).