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Worksheet N°1
The real numbers
LMD 1st year 2023-2024

**Exercise 1**

1. Let $r_1 = 0,2023$, $r_2 = 0,20232023...2023$ and $r_3 = 0,20232023...$
   Convert the decimal numbers $r_1$, $r_2$ and $r_3$ to fractions.
2. Let $a \in \mathbb{R}$, Show that, if $a^2$ is factor of 2 , then $a$ is also factor of 2.
3. Show that, the numbers $\sqrt{2}, \sqrt{3}, \sqrt{2}+\sqrt{3}$ are irrationals numbers.
4. Show that, if $n$ is prime number then $\sqrt{n}$ is irrational number.
5. Show that $\forall (p,q) \in \mathbb{N},\ p \neq q : (\sqrt{p}+\sqrt{q}) \notin \mathbb{Q} \Rightarrow (\sqrt{p}-\sqrt{q}) \notin \mathbb{Q}$.

**(optional)**

1. Let $n \in \mathbb{N}^*$, prove that, $\sqrt{\dfrac{n}{n+2}} \notin \mathbb{Q}$.
2. Let $n \in \mathbb{N}$, if $n$ is not a perfect square then $\sqrt{n} \notin \mathbb{Q}$.
3. Let $n, m \in \mathbb{N}$, which are not perfect squares, then $\sqrt{n}+\sqrt{m} \notin \mathbb{Q}$.
4. Show that, if $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$ then $x+y \notin \mathbb{Q}$.
5. Show that, if $x \in \mathbb{Q}^*$ and $y \notin \mathbb{Q}$ then $x \times y \notin \mathbb{Q}$.

**Exercise 2**

1. Resolve in $\mathbb{R}$: $|x-2|+|x-3|=3$, and $\left|2x^2-1\right| \leq |x+1|$
2. let $x$ and $y$ two real numbers, show that
   (a) $|x|+|y| \leq |x+y|+|x-y|$.
   (b) $1+|xy-1| \leq (1+|x-1|)(1+|y-1|)$.
   (c) $||x|-|y|| \leq |x-y| \leq |x|+|y|$. **(optional)**
   (d) For all $x \in \mathbb{R}$, we put $f(x) = \dfrac{|x|}{1+|x|}$.
   Show that, $\forall x, y \in \mathbb{R},\ f(x+y) \leq f(x)+f(y)$. **(optional)**

**Exercise 3**

1. Resolve in $\mathbb{R}$

$$\begin{array}{rcl} E(\frac{2x+1}{3})-2 & = & 0, \quad E(x+a)=2, a\in\mathbb{R}. \quad 3E(x^2+2x)-4=0, \\ E(x) & \geq & 1, \quad -1 \leq E(3x) \leq 1, \quad E(x)+|x-1|=x, \end{array}$$

2. Show that $\forall n \in \mathbb{N}^*$, we have $E(x+n) = E(x)+n$ and $E(\frac{1}{n}E(nx)) = E(x)$.

3. Let $x, y \in \mathbb{R}$, Show that $E(x) + E(y) \leq E(x+y) \leq E(x) + E(y) + 1$.

4. Prove that, $\forall p \in \mathbb{Z}$ , $E(p) + E(-p) = 0$. **(optional)**

5. Prove that, $\forall p \in \mathbb{R} \backslash \mathbb{Z}$, $E(p) + E(-p) + 1 = 0$. **(optional).**

**Exercise 4**

1. Give the definition of interval $I \subset \mathbb{R}$.

2. Let $A, B, C$ the subsets of $\mathbb{R}$, such that
   $A = \{x \in \mathbb{R},\ x^2 < 1\}$, $B = \{x \in \mathbb{R}, (x-3)(x+2) \geq 0\} \cap [-4, 4]$,
   $C = \{x \in \mathbb{R}^*,\ \dfrac{1}{x} > 2\}$.

   (a) Put these sets in the form of an interval of $\mathbb{R}$, or an interval union.

   (b) Why is the set $B$ not an interval?

   (c) Find the set of all upper bound, lower bound, supremum, infimum, maximum and minimum if there exists.

**Exercise 5**

1. Find the sets of all upper bound, lower bound, supremum, infimum, maximum and minimum if there exists of:
   $A = \left\{\frac{x+1}{x+2},\ x \in \mathbb{R}, x \leq -3\right\}$, $B = \left\{\frac{n+3}{n+2},\ n \in \mathbb{N}\right\}$, $C = \left\{(-1)^n + \frac{1}{n^2},\ n \in \mathbb{N}^*\right\}$.
   $D = \left\{2 - \frac{8}{n+4},\ n \in \mathbb{N}\right\}$, $E = \left\{\frac{(-1)^n}{n+1} + \frac{(-1)^n+2}{3}\ ,\ n \in \mathbb{N}\right\}$.
   **(D and E optional).**

2. For using the caracterization of supremum and infimum, prove that

$$\max(]0,2]) = \sup(]0,2]) = 2,\ \ \inf(]0,2]) = 0.$$

$$\sup B = \frac{3}{2},\ \ \inf B = 1$$

3. Same question for set $D$.**(optional)**

**Exercise 6**

Let $A$ and $B$ be nonempty subsets of real numbers, prove that

1. If $(A \subset B) \Rightarrow (\sup A \leq \sup B)$ et $(\inf B \leq \inf A)$.

2. $\sup(A \cup B) = \max\{\sup A,\ \sup B\}$.

3. $\inf(A \cup B) = \min\{\inf A,\ \inf B\}$. **(optional)**