Université Aboubekr BELKAID - Tlemcen	A.U 2021/2022 - 2ème année Mathématiques
Faculté des Sciences - Département de Mathématiques	Analyse 4 - Fiche de T.D n°3

Exercice 1 : Évaluer chacune des intégrales suivantes dans les domaines respectifs :

$$\iint_{\mathcal{D}_1} \frac{x^2}{1+y^2}\,dxdy \qquad \mathcal{D}_1 = \left\{(x,y) \in \mathbb{R}^2 \diagup 0 \leq x \leq 1\ ,\ 0 \leq y \leq 1\right\}$$

$$\iint_{\mathcal{D}_2} \frac{x^2}{y^2}\,dxdy \qquad \mathcal{D}_2 = \left\{(x,y) \in \mathbb{R}^2 \diagup 1 \leq x \leq 2\ ,\ 1 \leq xy \leq x^2\right\}$$

$$\iint_{\mathcal{D}_3} (x+y)e^{-(x+y)}\,dxdy \qquad \mathcal{D}_3 = \left\{(x,y) \in \mathbb{R}^2 \diagup x \geq 0\ ,\ y \geq 0\ ,\ x+y \leq 1\right\}$$

Exercice 2 : Pour chacune des intégrales doubles suivantes dessiner le domaine d'intégration, puis changer l'ordre d'intégration.

$$\int_1^3 \left(\int_{x^2}^{x+9} f(x,y)\,dy\right) dx \quad , \quad \int_0^4 \left(\int_y^{10-y} f(x,y)\,dx\right) dy$$

Exercice 3 : En passant en coordonnées polaires, calculer les intégrales suivantes :

$$I_1 = \iint_{\mathcal{D}_1} \frac{xy}{x^2+y^2}\,dxdy\ ,\mathcal{D}_1 = \left\{(x,y) \in \mathbb{R}^2 \diagup x \geq 0\ ,\ y \geq 0\ ,\ 1 \leq x^2+y^2 \leq 4\right\}$$

$$I_2 = \iint_{\mathcal{D}_2} \sqrt{x^2+y^2}\,dxdy\,,\mathcal{D}_2 = \left\{(x,y) \in \mathbb{R}^2 \diagup x \geq 0\ ,\ 1 \leq x^2+y^2 \leq 2y\right\}$$

Exercice 4 : Calculer l'aire de chacune des régions du plan définies par :

$$\mathcal{A} = \left\{(x,y) \in \mathbb{R}^2 \diagup x \geq 0\ ,\ x^3 \leq y \leq x^2\right\}$$
$$\mathcal{B} = \left\{(x,y) \in \mathbb{R}^2 \diagup y \leq 8\ ,\ x^2 \leq y \leq x^3\right\}$$
$$\mathcal{C} = \left\{(x,y) \in \mathbb{R}^2 \diagup x^2+y^2 \leq 2x+3\ ,\ x^2+y^2 \leq -2x+3\right\}$$

Exercice 5 : Évaluer les intégrales triples suivantes

$$I_1 = \iiint_{\mathcal{D}_1} \frac{dxdydz}{\sqrt{1+x+y+z}},\ \mathcal{D}_1 = [0,1]\times[0,1]\times[0,1]$$

$$I_2 = \iiint_{\mathcal{D}_2} z\,dxdydz,\ \mathcal{D}_2 = \left\{(x,y,z) \in (\mathbb{R}^+)^3 \diagup x^2+z \leq 1\,,\ y^2+z \leq 1\right\}$$

Exercice 6 : Calculer le volume de la région de $\mathbb{R}^3$ délimitée par la sphère $x^2+y^2+z^2 = 2Rz$, le cône $x^2+y^2 = z^2$ et contenant le point $(0,0,R)$.

2ème année de Licence de Maths - Semestre 4 - 2021/2022.

Module : « Analyse IV » - Série de T.D. N° 3 - Corrigé.

Ex1: * $I_1=\iint_{D_1} \frac{x^2}{1+y^2}\,dx\,dy$; $D_1=\{(x,y)\in\mathbb{R}^2 / 0\le x\le 1 ,\ 0\le y\le 1\}$.

$$I_1=\int_0^1\left(\int_0^1 \frac{x^2}{1+y^2}\,dx\right)dy = \left[\frac{x^3}{3}\right]_0^1\cdot\left[\operatorname{arctg} y\right]_0^1 = \frac{1}{3}\cdot\frac{\pi}{4} = \boxed{\frac{\pi}{12}}.$$

* $I_2=\iint_{D_2} \frac{x^2}{y^2}\,dx\,dy$; $D_2=\{(x,y)\in\mathbb{R}^2 / 1\le x\le 2 ;\ 1\le xy\le x^2\}$.

Comme $x>0$ car $1\le x\le 2$, alors $\frac{1}{x}\le y\le x$. Ainsi

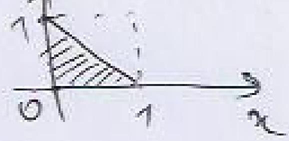
$$I_2=\int_1^2\left(\int_{1/x}^{x} \frac{dy}{y^2}\right)x^2\,dx = \int_1^2 \left[-\frac{1}{y}\right]_{1/x}^{x}\cdot x^2\,dx$$

$$I_2=\int_1^2\left(x-\frac{1}{x}\right)x^2\,dx = \int_1^2 (x^3-x)\,dx = \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_1^2$$

$$I_2=(4-2)-\left(\frac{1}{4}-\frac{1}{2}\right) = 2-\left(\frac{-1}{4}\right) = \boxed{9/4}.$$

* $I_3=\iint_{D_3} (x+y)\,e^{-(x+y)}\,dx\,dy$; $D_3=\{x\ge 0,\ y\ge 0,\ x+y\le 1\}$

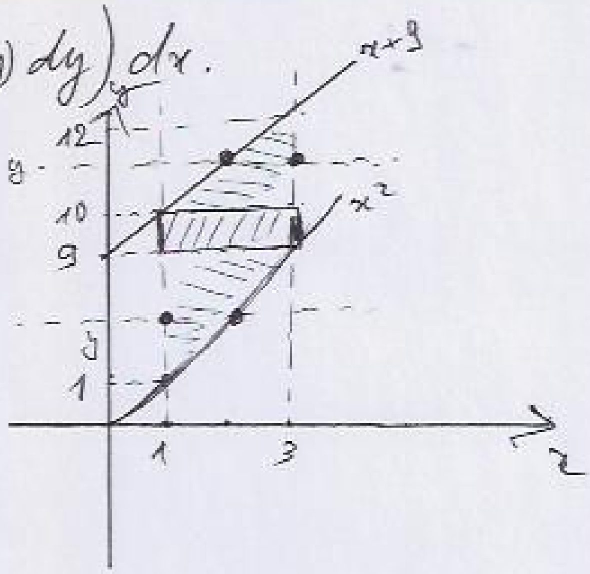
$$I_3=\int_0^1\left(\int_0^{1-x} (x+y)\,e^{-y}\,dy\right)e^{-x}\,dx$$



$$=\int_0^1 \left[-e^{-y}(x+y+1)\right]_0^{1-x}\cdot e^{-x}\,dx = \int_0^1 \left[(x+1)e^{-x}-2e^{-1}\right]dx.$$

$$=\left[-e^{-x}(x+2)\right]_0^1 - 2e^{-1} = 2-3e^{-1}-2e^{-1} = \boxed{2-5e^{-1}}$$

EX2: $\ast$ $A = \displaystyle\int_1^3 \left(\int_{x^2}^{x+9} f(x,y)\,dy \right) dx.$

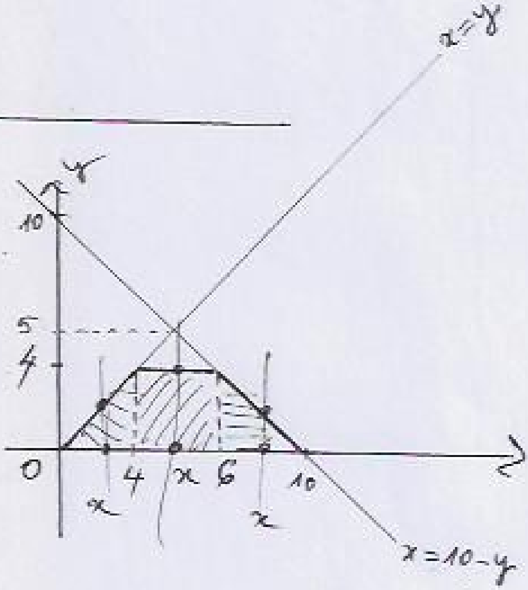


$$A = \int_1^9 \left(\int_1^{\sqrt{y}} f(x,y)\,dx \right) dy$$

$$+ \int_9^{10} \left(\int_1^3 f(x,y)\,dx \right) dy$$

$$+ \int_{10}^{12} \left(\int_{y-9}^3 f(x,y)\,dx \right) dy.$$

$\ast$ $B = \displaystyle\int_0^4 \left(\int_y^{10-y} f(x,y)\,dx \right) dy$

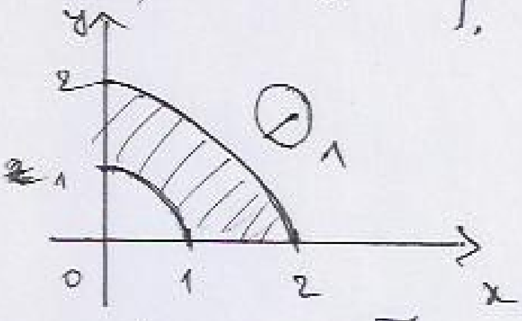


$$B = \int_0^4 \left(\int_0^{x} f(x,y)\,dy \right) dx$$

$$+ \int_4^6 \left(\int_0^4 f(x,y)\,dy \right) dx$$

$$+ \int_6^{10} \left(\int_0^{10-x} f(x,y)\,dy \right) dx.$$

Ex3 : * $I_1 = \iint_{D_1} \frac{xy}{x^2+y^2}\,dx\,dy$, $D_1 = \{x \geq 0, y \geq 0, 1 \leq x^2+y^2 \leq 4\}$.



$$\begin{cases} x = r\cos\theta & 1 \leq r \leq 2 \\ y = r\sin\theta & 0 \leq \theta \leq \pi/2 \end{cases}$$

$$I_1 = \int_1^2 \int_0^{\pi/2} \frac{r^2\cos\theta\sin\theta}{\cancel{r^2}}\, r\,dr\,d\theta = \left[\frac{r^2}{2}\right]_1^2 \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}$$

$$I_1 = \left(2 - \frac{1}{2}\right)\left(\frac{1}{2}\right) = \boxed{3/4}$$

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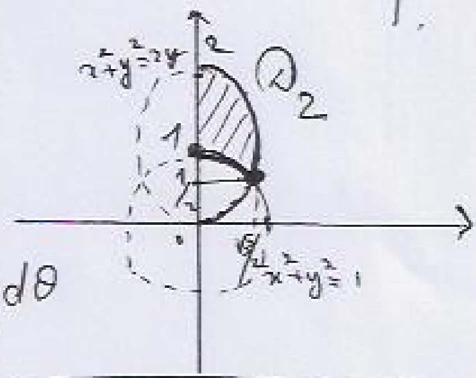
* $I_2 = \iint_{D_2} \sqrt{x^2+y^2}\,dx\,dy$; $D_2 = \{x \geq 0, 1 \leq x^2+y^2 \leq 2y\}$.



$$\begin{cases} x = r\cos\theta, & 1 \leq r \leq 2\sin\theta \\ y = r\sin\theta & \frac{\pi}{6} \leq \theta \leq \pi/2 \end{cases}$$

$$I_2 = \int_{\pi/6}^{\pi/2} \left(\int_1^{2\sin\theta} r^2\,dr \right) d\theta = \frac{1}{3}\int_{\pi/6}^{\pi/2} (8\sin^3\theta - 1)\,d\theta$$

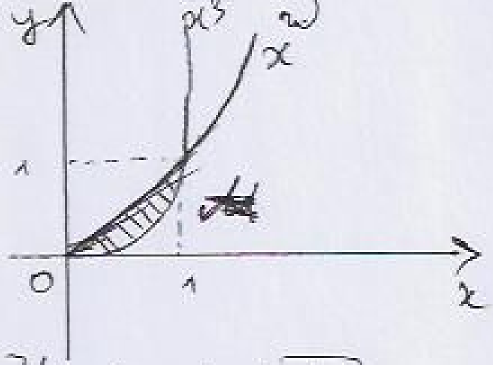
Il faut linéariser $\sin^3\theta$: $\boxed{8\sin^3\theta = 6\sin\theta - 2\sin 3\theta}$

$$I_2 = \frac{1}{3}\left[-6\cos\theta + \frac{2}{3}\cos 3\theta - \theta\right]_{\pi/6}^{\pi/2}$$

$$= \frac{1}{3}\left[\left(0 + 0 - \frac{\pi}{2}\right) - \left(-6\cos\pi/6 + \frac{2}{3}\cancel{\cos\pi/2} - \pi/6\right)\right]$$

$$\boxed{I_2 = \sqrt{3} - \frac{\pi}{9}}$$

Ex4: * $\mathcal{A} = \left\{ (x,y) \in \mathbb{R}^2 \,/\, x \geq 0 \,;\, x^3 \leq y \leq x^2 \right\}$

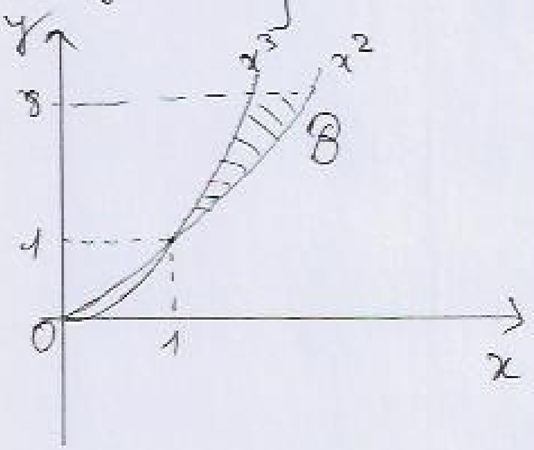


$$\text{Aire}(\mathcal{A}) = \iint_{\mathcal{A}} dx\,dy$$

$$= \int_0^1 \left(\int_{x^3}^{x^2} dy \right) dx$$

$$= \int_0^1 \left(x^2 - x^3 \right) dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$$

* $\mathcal{B} = \left\{ (x,y) \in \mathbb{R}^2 \,/\, y \leq 8 \,;\, x^2 \leq y \leq x^3 \right\}$



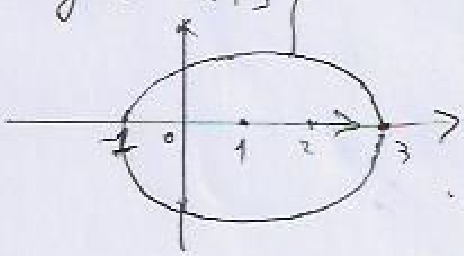
$$\text{Aire}(\mathcal{B}) = \iint_{\mathcal{B}} dx\,dy$$

$$= \int_1^8 \left(\int_{y^{1/3}}^{y^{1/2}} dx \right) dy$$

$$= \int_1^8 \left(y^{1/2} - y^{1/3} \right) dy = \left[\frac{2}{3} y^{3/2} - \frac{3}{4} y^{4/3} \right]_1^8$$

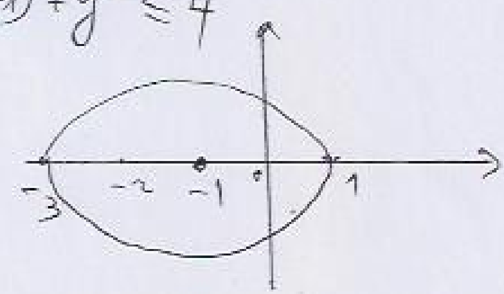
$$= \left(\frac{2}{3} \cdot 8\sqrt{8} - \frac{3}{4} 2^4 \right) - \left(\frac{2}{3} - \frac{3}{4} \right) = \frac{32}{3}\sqrt{2} - 12 - \frac{2}{3} + \frac{3}{4} = \boxed{\frac{32}{3}\sqrt{2} - 143}$$

* $\mathcal{C} = \left\{ (x,y) \in \mathbb{R}^2 \,/\, x^2 + y^2 \leq 2x + 3 \,;\, x^2 + y^2 \leq -2x + 3 \right\}$

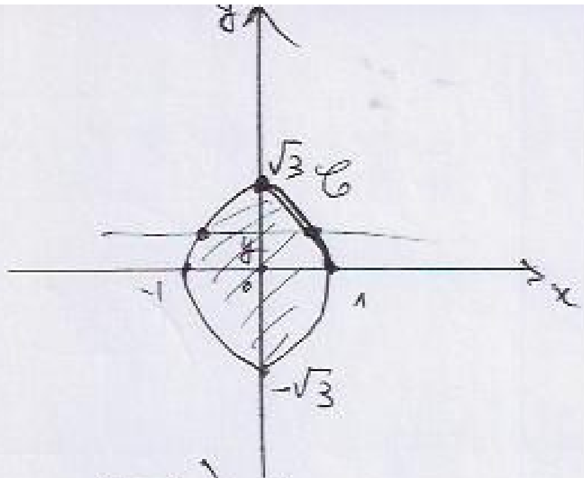


. $x^2 + y^2 \leq 2x + 3 \iff (x-1)^2 + y^2 \leq 4$

. $x^2 + y^2 \leq -2x + 3 \iff (x+1)^2 + y^2 \leq 4$



$$\text{Aire}(\mathcal{C}) = \iint_{\mathcal{C}} dx\,dy$$



$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(\int_{1-\sqrt{4-y^2}}^{-1+\sqrt{4-y^2}} dx \right) dy$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(-2+2\sqrt{4-y^2}\right) dy = 4\int_{0}^{\sqrt{3}} \left(-1+2\sqrt{1-\left(\tfrac{y}{2}\right)^2}\right) dy$$

Posons $\boxed{y = 2\sin\theta}$ $\quad 0 \leq \theta \leq \pi/3$

$$\text{Aire}(\mathcal{C}) = 8\int_0^{\pi/3} (-1+2\cos\theta)\cos\theta \, d\theta.$$

$$= 8\int_0^{\pi/3} \left(-\cos\theta + \cos(2\theta) + 1\right) d\theta.$$

$$= 8\left[\theta - \sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/3}$$

$$= 8\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right) = \boxed{8\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)}.$$

Exercice 5: $\quad I_1 = \iiint_{D_1} \dfrac{dx\,dy\,dz}{\sqrt{1+x+y+z}}, \quad D_1 = [0,1]\times[0,1]\times[0,1]$

$$I_1 = \int_0^1\int_0^1 \left[2\sqrt{1+x+y+z}\right]_{z=0}^{z=1} dx\,dy = 2\int_0^1\int_0^1 \left(\sqrt{2+x+y} - \sqrt{1+x+y}\right) dx\,dy$$

$$= 2\int_0^1 \left[\frac{2}{3}(2+x+y)^{3/2} - \frac{2}{3}(1+x+y)^{3/2}\right]_{y=0}^{y=1} dx.$$

$$= \frac{4}{3}\int_0^1 \left((3+x)^{3/2} - (2+x)^{3/2} - (2+x)^{3/2} + (1+x)^{3/2}\right) dx$$

$$= \frac{4}{3}\left[\frac{2}{5}(3+x)^{5/2} - \frac{4}{5}(2+x)^{5/2} + \frac{2}{5}(1+x)^{5/2}\right]_0^1 = \frac{8}{15}\left[4^{5/2} - 2\cdot 3^{5/2} + 2^{5/2} - 3^{5/2} + 2\cdot 2^{5/2} - 1\right]$$

$$\boxed{I_1 = \frac{8}{15}\left(31 - 27\sqrt{3} + 12\sqrt{2}\right)}$$

$$I_2 = \iiint_{D_2} z \, dx\,dy\,dz \; ; \; D_2 = \left\{ (x,y,z) \in \mathbb{R}_+^3 \,/\, x^2+z \leq 1 \; ; y^2+z \leq 1 \right\}.$$

On a $z \geq 0$ et $z \leq 1-x^2$, $z \leq 1-y^2 \iff 0 \leq z \leq \min(1-x^2, 1-y^2)$

$$I_2 = \int_0^1 \int_0^1 \left[\frac{z^2}{2} \right]_0^{\min(1-x^2,1-y^2)} dx\,dy$$

$$= \frac{1}{2} \int_0^1 \int_0^1 \min^2(1-x^2, 1-y^2)\, dx\,dy = \iint_{0 \leq x \leq y \leq 1} (1-y^2)\, dx\,dy$$

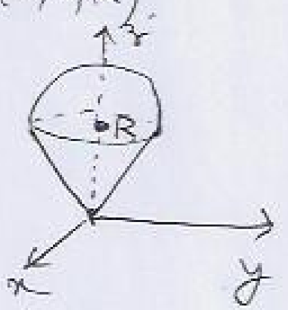
$$I_2 = \int_0^1 \left(\int_0^y dx \right) (1-y^2)\, dy = \int_0^1 y(1-y^2)\, dy$$

$$I_2 = \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = \frac{1}{2} - \frac{1}{4} = \boxed{\frac{1}{4}}$$

Ex6 : S la région de $\mathbb{R}^3$ délimitée par la sphère $x^2+y^2+z^2 = 2Rz$ et le cône $x^2+y^2 = z^2$ et contenant le pt $(0,0,R)$.

$x^2+y^2+z^2 = 2Rz \iff x^2+y^2+(z-R)^2 = R^2$

S est définie par les inégalités $\begin{cases} x^2+y^2+(z-R)^2 \leq R^2 \\ x^2+y^2 \leq z^2 \end{cases}$



$\iff \sqrt{x^2+y^2} \leq z \leq R + \sqrt{R^2-(x^2+y^2)}$

$$Vol(S) = \iiint_S dx\,dy\,dz = \iint_{x^2+y^2 \leq R^2} \left(R + \sqrt{R^2-(x^2+y^2)} - \sqrt{x^2+y^2} \right) dx\,dy$$

$$Vol(S) = \int_0^{2\pi} \int_0^R \left[R + \sqrt{R^2-r^2} - r \right] r\,dr\,d\theta = 2\pi \left[R\frac{r^2}{2} - \frac{r^3}{3} - \frac{1}{3}(R^2-r^2)^{3/2} \right]_0^R$$

$$Vol(S) = 2\pi \left[\frac{R^3}{2} - \cancel{\frac{R^3}{3}} + \cancel{\frac{R^3}{3}} \right]$$

$$\boxed{Vol(S) = \pi R^3}$$