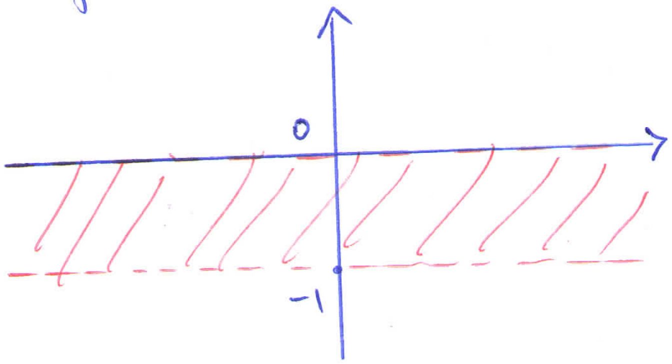
Solution TD2 : Exercices facultatifs

Ex6 : (1) $0 < \mathrm{Re}(iz) < 2$. On a :

$iz = i(x+iy) = ix - y$ donc $0 < -y < 1$

$\Rightarrow -1 < y < 0$

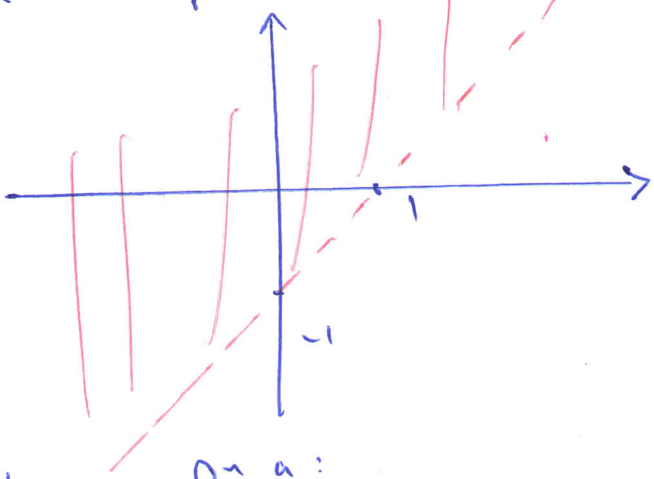
L'ensemble vérifiant (1) est une bande.



(2) $\mathrm{Re}\, z - \mathrm{Im}\, z < 1$

$\Rightarrow x - y < 1$.

Ainsi, l'ensemble des points est un demi-plan.



(3) $|\bar{z} - 4 + i| = 1$. On a :

$|\bar{z} - 4 + i| = |\overline{z - 4 + i}| = |z - 4 - i| = |z - (4+i)| = 1$

et ceci est un cercle de centre $4+i$ et de rayon 1.

(4) $\mathrm{Re}\left(\frac{1}{z}\right) = 2$. On a :

$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$.

$$\mathrm{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2} = 2 \Rightarrow x^2+y^2-\frac{x}{2}=0$$

$$\Rightarrow \left(x-\frac{1}{4}\right)^2 + y^2 = \frac{1}{16}.$$

Un cercle de centre $\left(\frac{1}{4},0\right)$ et de rayon $\frac{1}{4}$ privé du point $(0,0)$.

(5) $|z-i| = 3$ est un cercle de centre i et de rayon 3.

(6) $\left|\frac{z-3}{z-5}\right| = \frac{\sqrt{2}}{2}$. On a :

$2|z-3| = \sqrt{2}|z-5| \Rightarrow 4\left((x-3)^2+y^2\right) = 2\left((x-5)^2+y^2\right)$

$$\Rightarrow 2(x-3)^2 + 2y^2 - (x-5)^2 - y^2 = 0$$

$$\Rightarrow x^2 - 2x - 7 + y^2 = 0$$

$\Rightarrow (x-1)^2 + y^2 = 8$ qui est un cercle de centre $(1,0)$ et de rayon $\sqrt{8}$.

Exo7 : 1) $z^3 = 1 = 1(\cos 0 + i\sin 0)$.

Donc, les racines de l'équation $z^3=1$ sont

$$z_k = \cos\left(\frac{2\pi k}{3}\right) + i\sin\left(\frac{2\pi k}{3}\right), \quad k = 0,1,2.$$

C.à.d $z_0 = 1$

$$z_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = e^{\frac{2\pi i}{3}}$$

et $z_2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = e^{\frac{4\pi i}{3}}$

2) puisque $\sin\left(\frac{2\pi}{3}\right) = 2\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

et $\sin\left(\frac{4\pi}{3}\right) = 2\cos\left(\frac{2\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\left(\cos\frac{2\pi}{3} = \cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3} = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}\right)$, alors

On déduit que $j = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$.

a) $j^{2} = e^{4\pi i/3}$ qui est une solution.

b) On a $j^3 = 1 \Rightarrow j^2 j = 1 \Rightarrow j^2 = \frac{1}{j} = e^{-2\pi i/3} = \bar{j}$.

c) $1 + j + j^2 = 1 + j + \bar{j} = 1 + 2\cos\left(\frac{2\pi}{3}\right) = 1 + 2\left(-\frac{1}{2}\right) = 0$.

EXO 8 : Remarquons que

si $z = \bar{z}$, alors $z \in \mathbb{R}$.

Ainsi,

$$\overline{i\left(\frac{z+1}{z-1}\right)} = -i\left(\frac{\bar{z}+1}{\bar{z}-1}\right) = -i\left(\frac{\frac{1}{z}+1}{\frac{1}{z}-1}\right) = i\left(\frac{z+1}{z-1}\right).$$

EXO 9 : On a : $z = \cos^2\theta + i\sin\theta\cos\theta$.

1) $z = 0$ implique que $\cos^2\theta + i\sin\theta\cos\theta = 0$

$\Rightarrow \cos\theta(\cos\theta + i\sin\theta) = 0$

$\Rightarrow \cos\theta = 0$ ou $(\cos\theta + i\sin\theta) = 0$ (qui est impossible).

Ainsi, $\theta = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$

2) On sait que $z = \cos\theta\, e^{i\theta}$. Donc, on obtient

$$z^{-1} = \frac{e^{-i\theta}}{\cos\theta} = 1 - i\operatorname{tg}\theta.$$

$$z^2 = \cos^2\theta\, e^{2i\theta} = \cos^2\theta(\cos^2\theta - \sin^2\theta) + 2i\sin\theta\cos^3\theta$$

$$z^3 = \cos^3\theta\, e^{3i\theta} = \cos^3\theta(\cos(3\theta) + i\sin(3\theta))$$

et $$z^{-3} = \frac{e^{-3i\theta}}{\cos^3\theta} = \frac{\cos(3\theta)}{\cos^3\theta} - i\,\frac{\sin(3\theta)}{\cos^3\theta}.$$

Ex 10 : Soit $z = x+iy$. Alors

$$\frac{z-1}{z+1} = \frac{x-1+iy}{x+1+iy} = \frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)}$$

$$= \frac{(x-1)(x+1)+iy(x+1)-iy(x-1)+y^2}{(x+1)^2+y^2}$$

$$= \frac{x^2+y^2-1+2iy}{(x+1)^2+y^2} = \frac{x^2+y^2-1}{(x+1)^2+y^2} + i\,\frac{2y}{(x+1)^2+y^2}.$$

puisque $|z|=1 \Rightarrow x^2+y^2=1 \Rightarrow x^2+y^2-1=0$.

Ainsi $\operatorname{tg}\theta = \frac{2y}{0} = \begin{cases} +\infty & \text{si } y>0 \\ -\infty & \text{si } y<0. \end{cases}$

Donc $\theta = \begin{cases} \pi/2 & \text{si } y>0 \\ -\pi/2 & \text{si } y<0. \end{cases}$

Ex 11 : 1) Remarquons que

$$\sum_{k=0}^{4} e^{ik\pi/5} = \sum_{k=0}^{4} \left(e^{i\pi/5}\right)^k = \frac{1-e^{i5\pi/5}}{1-e^{i\pi/5}} = \frac{1-e^{i\pi}}{1-e^{i\pi/5}} = \frac{2}{1-e^{i\pi/5}}.$$

2) $$\sum_{k=0}^{4} e^{ik\pi/5} = \sum_{k=0}^{4} \cos\left(\frac{k\pi}{5}\right) + i\sin\left(\frac{k\pi}{5}\right).$$

D'autre part,

$$\sum_{k=0}^{4} e^{ik\frac{\pi}{5}} = \frac{2}{1-e^{i\pi/5}} = \frac{2}{1-\cos\frac{\pi}{5}-i\sin\frac{\pi}{5}}$$

$$= \frac{2\left(1-\cos\frac{\pi}{5}+i\sin\left(\frac{\pi}{5}\right)\right)}{\left(1-\cos\frac{\pi}{5}\right)^2+\sin^2\frac{\pi}{5}} = \frac{2\left(1-\cos\frac{\pi}{5}\right)+2i\sin\left(\frac{\pi}{5}\right)}{2-2\cos\left(\frac{\pi}{5}\right)}$$

Ainsi, $\displaystyle\sum_{k=0}^{4} \cos\left(\frac{k\pi}{5}\right) = 1$ et

$$\sum_{k=0}^{4} \sin\left(\frac{k\pi}{5}\right) = \frac{\sin(\pi/5)}{1-\cos\left(\frac{\pi}{5}\right)}.$$

Ex 12 : $|z|=1 \iff \bar{z} = \dfrac{1}{z} \iff \overline{\left(\dfrac{z_1-z_2}{1-z_1\bar{z}_2}\right)} = \dfrac{1-z_1\bar{z}_2}{z_1-z_2}$

$$\iff \frac{\bar{z}_1-\bar{z}_2}{1-\bar{z}_1 z_2} = \frac{1-z_1\bar{z}_2}{z_1-z_2}$$

$$\iff (\bar{z}_1-\bar{z}_2)(z_1-z_2) = (1-z_1\bar{z}_2)(1-\bar{z}_1 z_2)$$

or $(\bar{z}_1-\bar{z}_2)(z_1-z_2) = \bar{z}_1 z_1 - \bar{z}_2 z_1 - z_2\bar{z}_1 + \bar{z}_2 z_2$

$$= |z_1|^2+|z_2|^2-(z_1\bar{z}_2+z_2\bar{z}_1)$$

et $(1-z_1\bar{z}_2)(1-\bar{z}_1 z_2) = 1 - z_2\bar{z}_1 - z_1\bar{z}_2 + z_1\bar{z}_2\bar{z}_1 z_2$

$$= 1+|z_1|^2|z_2|^2-(z_2\bar{z}_1+z_1\bar{z}_2).$$

Ainsi,

$|z|=1 \iff |z_1|^2+|z_2|^2 = 1+|z_1|^2|z_2|^2$

$\iff 1+|z_1|^2|z_2|^2 - |z_1|^2|z_2|^2 = 0$

$\iff (|z_1|^2-1)(|z_2|^2-1) = 0$

$\iff |z_1|=1$ ou $|z_2|=1$. $\blacksquare$