# Solution TD 2 : Intégrales et calcul des primitives

## I) Intégrales indéfinies :

**Ex 1 :** (1) $$\int \frac{x^2+1}{\sqrt[3]{x}}\,dx = \int (x^2+1)\,x^{-1/3}\,dx$$

$$= \int \left(x^{5/3} + x^{-1/3}\right) dx$$

$$= \frac{1}{\frac{5}{3}+1}\,x^{\frac{5}{3}+1} + \frac{1}{-\frac{1}{3}+1}\,x^{-\frac{1}{3}+1} + C, \quad C \in \mathbb{R}$$

$$= \frac{3}{8}\,x^{8/3} + \frac{3}{2}\,x^{2/3} + C, \quad C \in \mathbb{R}.$$

(2) $$\int \frac{(\ln x)^2}{x}\,dx = \int \frac{1}{x}(\ln x)^2\,dx = \frac{(\ln x)^3}{3} + C, \quad C \in \mathbb{R}$$

(On peut poser $u = \ln x$, on obtient $\int u'(x)\,u^2(x)\,dx = \frac{u^3(x)}{3} + C$).

(3) $$\int \cos(3x+2)\,dx = \frac{1}{3}\int 3\cos(3x+2)\,dx$$

On pose $u(x) = 3x+2$, alors on obtient

$$\frac{1}{3}\int 3\cos(3x+2)\,dx = \frac{1}{3}\int u'(x)\cos(u(x))\,dx$$

$$= \frac{1}{3}\sin(u(x)) + C, \quad C \in \mathbb{R}$$

$$= \frac{1}{3}\sin(3x+2) + C.$$

(4) $$\int (\operatorname{sh} x)^2\,dx = \int \left(\frac{e^x - e^{-x}}{2}\right)^2 dx = \frac{1}{4}\int \left(e^{2x} + e^{-2x} - 2\right) dx$$

$$= \frac{1}{4}\left[\frac{e^{2x}}{2} + \frac{e^{-2x}}{-2} - 2x\right] + C, \quad C \in \mathbb{R}$$

$$= \frac{1}{4}\left[\frac{e^{2x} - e^{-2x}}{2} - 2x\right] + C = \frac{1}{4}\operatorname{sh}(2x) - \frac{x}{2} + C, \quad C \in \mathbb{R}.$$

5) $\displaystyle\int \frac{e^x}{e^{-x}+e^x}\,dx = \int \frac{e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{1+e^{2x}}\,dx$

On pose $u = 1+e^{2x}$, on obtient

$$\frac{1}{2}\int \frac{u'(x)}{u(x)}\,dx = \frac{1}{2}\ln|u(x)| + c \quad , \quad c \in \mathbb{R}$$

$$= \frac{1}{2}\ln(1+e^{2x}) + c .$$

Ex2: (1) $\displaystyle\int \frac{\ln x}{x^n}\,dx$ , $n \neq 1$.

On sait que $\displaystyle\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx$.

Donc, si on pose

$u(x) = \ln x$ , $u'(x) = \frac{1}{x}$

$v'(x) = \frac{1}{x^n} = x^{-n}$ , $v(x) = \frac{x^{-n+1}}{-n+1}$ , alors

$$\int \frac{\ln x}{x^n}\,dx = \frac{x^{1-n}\ln x}{1-n} - \int \frac{x^{-n}}{1-n}\,dx$$

$$= \frac{x^{1-n}\ln x}{1-n} + \frac{1}{n-1}\int x^{-n}\,dx = \frac{x^{1-n}}{1-n}\ln x + \frac{1}{n-1}\left[\frac{x^{-n+1}}{-n+1}\right] + c \, , \quad c \in \mathbb{R}$$

$$= \frac{x^{1-n}\ln x}{1-n} - \frac{x^{-n+1}}{(n-1)^2} + c .$$

(2) $\displaystyle\int (x^2-x)\sin x\,dx = ?$

On pose

$u(x) = x^2 - x$ , $u'(x) = 2x-1$

$v'(x) = \sin x$ , $v(x) = -\cos x$

Ainsi, $\int (x^2-x)\sin x\,dx = -(x^2-x)\cos x + \int (2x-1)\cos x\,dx$

Une deuxième intégration par parties est nécessaire, pour cela,

On pose

$u(x) = 2x-1$ ; $u'(x) = 2$

$v'(x) = \cos x$ ; $v(x) = \sin x$

On a : $\int (2x-1)\cos x\,dx = (2x-1)\sin x - 2\int \sin x\,dx$

$= (2x-1)\sin x + 2\cos x + C$, $C \in \mathbb{R}$.

Donc, finalement,

$\int (x^2-x)\sin x\,dx = -(x^2-x)\cos x + (2x-1)\sin x + 2\cos x + C.$

(3) $\int e^x \cos x\,dx = ?$

On pose

$u(x) = e^x$ ; $u'(x) = e^x$

$v'(x) = \cos x$ ; $v(x) = \sin x$

$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$

Une deuxième intégration par parties donne :

$u(x) = e^x$ ; $u'(x) = e^x$

$v'(x) = \sin x$ ; $v(x) = -\cos x$

Donc,

$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$

Ainsi,

$$\int e^x \cos x\,dx = e^x \sin x - \left[ -e^x \cos x + \int e^x \cos x\,dx \right]$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

Ceci implique que

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$
$$\Rightarrow \int e^x \cos x\,dx = \frac{e^x}{2}\left( \sin x + \cos x \right) + C, \quad C \in \mathbb{R}.$$

Ex3: ① $\displaystyle\int \frac{dx}{x^2+x+1} = ?$

On a : $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1\right]$

On pose
$u = \dfrac{2x+1}{\sqrt{3}} \Rightarrow du = \dfrac{2}{\sqrt{3}}\,dx \Rightarrow dx = \dfrac{\sqrt{3}}{2}\,du.$

Ainsi,

$$\int \frac{dx}{x^2+x+1} = \int \frac{\frac{\sqrt{3}}{2}\,du}{\frac{3}{4}(u^2+1)} = \frac{2}{\sqrt{3}} \operatorname{arctg} u + c, \quad c \in \mathbb{R}$$
$$= \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + c.$$

② $\displaystyle\int \frac{dx}{x^2+x\sqrt{2x-x^2}} = ?$ ; $x \in ]0,2]$

On pose $x = \dfrac{1}{u} \Rightarrow dx = -\dfrac{1}{u^2}\,du$

$$\int \frac{dx}{x^2+x\sqrt{2x-x^2}} = \int \frac{-\frac{1}{u^2}\,du}{\frac{1}{u^2}+\frac{1}{u}\sqrt{\frac{2}{u}-\frac{1}{u^2}}} = \int \frac{-\frac{1}{u^2}\,du}{\frac{1}{u^2}+\frac{1}{u}\sqrt{\frac{2u^2-u}{u^3}}}$$

Ainsi,

$$\int \frac{dx}{x^2+x\sqrt{2x-x^2}} = \int \frac{-\frac{1}{u^2}\,du}{\frac{1}{u^2}+\frac{1}{u^2}\sqrt{\frac{2u^2-u}{u}}} = \int \frac{-du}{1+\sqrt{2u-1}}$$

On pose $t=\sqrt{2u-1} \Rightarrow dt = \frac{du}{\sqrt{2u-1}}$

Donc

$$\int \frac{-du}{1+\sqrt{2u-1}} = \int -\frac{\sqrt{2u-1}}{1+t}\,dt = -\int \frac{t\,dt}{1+t}$$

$$= -\int \frac{t+1-1}{t+1}\,dt = -\left[\int dt - \int \frac{dt}{1+t}\right]$$

$$= -t + \ln|1+t| + c, \quad c\in\mathbb{R}.$$

$$= -\sqrt{2u-1} + \ln\left(1+\sqrt{2u-1}\right) + c$$

$$= -\sqrt{\frac{2}{x}-1} + \ln\left(1+\sqrt{\frac{2}{x}-1}\right) + c.$$

③ $\int \sqrt{a^2+x^2}\,dx = ?$

. Si $a=0$, alors on a :

$$\int \sqrt{x^2}\,dx = \int |x|\,dx = -\frac{x|x|}{2} + c, \quad c\in\mathbb{R}$$

. Si $a\neq 0$, alors

$$\int \sqrt{a^2+x^2}\,dx = |a| \int \sqrt{1+\left(\frac{x}{a}\right)^2}\,dx$$

posons $\frac{x}{a} = \operatorname{sh} t \Rightarrow x = a \operatorname{sh} t \Rightarrow dx = a \operatorname{ch} t\, dt$

Ainsi, $\int \sqrt{a^2+x^2}\,dx = a|a| \int \sqrt{1+\operatorname{sh}^2 t}\ \operatorname{ch} t\, dt$

$= a|a| \int \sqrt{\operatorname{ch}^2 t}\ \operatorname{ch} t\ dt$

$= a|a| \int \operatorname{ch}^2 t\, dt \quad ; \quad \operatorname{ch} t \geq 0,\ \forall t \in \mathbb{R}$

$= \frac{a|a|}{2} \int (\operatorname{ch}(2t)+1)\, dt$

$= \frac{a|a|}{2} \left[ \frac{1}{2} \operatorname{sh}(2t) + t \right] + C \quad , \ C \in \mathbb{R}$

$= \frac{a|a|}{2} \left( \operatorname{sh} t \operatorname{ch} t + t \right) + C$

$= \frac{a|a|}{2} \left( \operatorname{sh} t \sqrt{1+\operatorname{sh}^2 t} + t \right) + C$

$= \frac{a|a|}{2} \left[ \frac{x}{a} \sqrt{1+\left(\frac{x}{a}\right)^2} + \operatorname{argsh}\left(\frac{x}{a}\right) \right] + C$

Ex4: (1) $\int \frac{x^2}{x^2-1}\, dx = ?$

$\frac{x^2}{x^2-1} = \frac{x^2-1+1}{x^2-1} = 1 + \frac{1}{x^2-1}$ et

$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{Ax+A+Bx-B}{x^2-1}$

$= \frac{(A+B)x + A-B}{x^2-1}$.

Par identification

$\begin{cases} A+B = 0 \\ A-B = 1 \end{cases} \Rightarrow \begin{cases} A = 1/2 \\ B = -1/2 \end{cases}$.

Ainsi,

$$\int \frac{x^2}{x^2-1}dx = \int dx + \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1}$$

$$= x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \quad , \quad C \in \mathbb{R}.$$

(2) $\displaystyle\int \frac{2dx}{x(x^2+1)} = ?$

$$\frac{2}{x(x^2+1)} = \frac{a}{x} + \frac{bx+c}{x^2+1} = \frac{ax^2+a+bx^2+cx}{x(x^2+1)}.$$

par identification

$$\begin{cases} a+b=0 \\ a=2 \\ c=0 \end{cases} \Rightarrow \begin{cases} a=2 \\ b=-2 \\ c=0 \end{cases}$$

Ainsi,

$$\int \frac{2dx}{x(x^2+1)} = \int \frac{2dx}{x} - \int \frac{2x\,dx}{x^2+1}$$

$$= 2\ln|x| - \ln(x^2+1) + C \quad , \quad C \in \mathbb{R}$$

(3) $\displaystyle\int \frac{x^3\,dx}{x^2+1} = ?$

Remarquons que $\displaystyle\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$.

Donc

$$\int \frac{x^3}{x^2+1}dx = \int\left(x - \frac{x}{x^2+1}\right)dx = \int x\,dx - \int \frac{1}{2}\,\frac{2x}{x^2+1}\,dx$$

$$= \frac{x^2}{2} - \frac{1}{2}\ln(x^2+1) + C, \quad C \in \mathbb{R}.$$

**Ex:** ① $\int \cos^3 x \, dx = ?$

On a : $\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$

posons $t = \sin x \Rightarrow dt = \cos x \, dx$.

Ainsi, $\int \cos^3 x \, dx = \int (1 - t^2) \, dt = t - \frac{t^3}{3} + c$, $c \in \mathbb{R}$

$= \sin x - \frac{1}{3} \sin^3 x + c.$

② $\int \cos^2 x \sin^3 x \, dx = \int \cos^2 x \sin^2 x \sin x \, dx$

$= \int \cos^2 x (1 - \cos^2 x) \sin x \, dx$

$= \int (\cos^2 x - \cos^4 x) \sin x \, dx$

posons $t = \cos x \Rightarrow dt = -\sin x \, dx$.

Ainsi,

$\int \cos^2 x \sin^3 x \, dx = -\int (t^2 - t^4) \, dt = \int (t^4 - t^2) \, dt$

$= \frac{t^5}{5} - \frac{t^3}{3} + c$, $c \in \mathbb{R}$

$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c.$

③ $\int \sin^2\left(\frac{x}{2}\right) \cos\left(\frac{3x}{2}\right) dx = ?$

On a :

$\int \sin^2\left(\frac{x}{2}\right) \cos\left(\frac{3x}{2}\right) dx = \int \left(\frac{1 - \cos x}{2}\right) \cos\left(\frac{3x}{2}\right) dx$

Ainsi,

$$\int\left(\frac{1-\cos x}{2}\right)\cos\left(\frac{3x}{2}\right)dx = \frac{1}{2}\int\cos\left(\frac{3x}{2}\right)dx - \frac{1}{2}\int\cos x\cos\left(\frac{3x}{2}\right)dx$$

$$= \frac{1}{2}\cdot\frac{2}{3}\sin\left(\frac{3x}{2}\right) - \frac{1}{2}\int\cos x\cos\left(\frac{3x}{2}\right)dx$$

Sachant que

$$\cos a\cos b = \frac{1}{2}\left(\cos(a+b)+\cos(a-b)\right)$$ , alors

$$\cos x\cos\left(\frac{3x}{2}\right) = \frac{1}{2}\left(\cos\left(\frac{5x}{2}\right)+\cos\left(-\frac{x}{2}\right)\right)$$

$$= \frac{1}{2}\left(\cos\left(\frac{5x}{2}\right)+\cos\left(\frac{x}{2}\right)\right).$$

Donc

$$\int\cos x\cos\left(\frac{3x}{2}\right)dx = \frac{1}{2}\int\left[\cos\left(\frac{5x}{2}\right)+\cos\left(\frac{x}{2}\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{2}{5}\sin\left(\frac{5x}{2}\right)+2\sin\left(\frac{x}{2}\right)\right]+c, \quad c\in\mathbb{R}.$$

$$= \frac{1}{5}\sin\left(\frac{5x}{2}\right)+\sin\left(\frac{x}{2}\right)+c.$$

Finalement,

$$\int\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right)dx = \frac{1}{3}\sin\left(\frac{3x}{2}\right) - \frac{1}{10}\sin\left(\frac{5x}{2}\right) - \frac{1}{2}\sin\left(\frac{x}{2}\right)+C.$$

Ex 6 : ① $\displaystyle\int\frac{dx}{\sin x} = ?$ posons $u = tg\left(\frac{x}{2}\right)$

$$\Rightarrow \begin{cases} dx = \dfrac{2\,du}{1+u^2} \\ \cos x = \dfrac{1-u^2}{1+u^2} \\ \sin x = \dfrac{2u}{1+u^2} \end{cases}$$

Ainsi,

$$\int\frac{dx}{\sin x} = \int\frac{\frac{2\,du}{1+u^2}}{\frac{2u}{1+u^2}} = \int\frac{du}{u} = \ln|u|+C, \quad C\in\mathbb{R}$$

Donc,

$$\int \frac{dx}{\sin x} = \ln\left|tg\left(\frac{x}{2}\right)\right| + c.$$

(2) $\displaystyle\int \frac{dx}{2+\cos x} = ?$

$$\int \frac{dx}{2+\cos x} = \int \frac{\frac{2\,du}{1+u^2}}{2+\frac{1-u^2}{1+u^2}} = \int \frac{2\,du}{u^2+3}$$

$$= \frac{2}{3}\int \frac{du}{\left(\frac{u}{\sqrt{3}}\right)^2+1} = \frac{2\sqrt{3}}{3}\int \frac{\frac{du}{\sqrt{3}}}{\left(\frac{u}{\sqrt{3}}\right)^2+1}$$

$$= \frac{2\sqrt{3}}{3}\, arctg\left(\frac{u}{\sqrt{3}}\right) + c, \quad c \in \mathbb{R}$$

$$= \frac{2\sqrt{3}}{3}\, arctg\left(\frac{1}{\sqrt{3}}\, tg\left(\frac{x}{2}\right)\right) + c$$

## II) Intégrales définies :

Exg : (1) $\displaystyle\int_1^2 \frac{\sqrt{x-1}}{x}\,dx = ?$

posons $t = \sqrt{x-1}$, $dt = \dfrac{dx}{2\sqrt{x-1}} = \dfrac{dx}{2t}$

Si $x=1$, alors $t=0$

Si $x=2$, alors $t=1$. Aussi, $t=\sqrt{x-1} \Rightarrow x = t^2+1$.

Ainsi,

$$\int_1^2 \frac{\sqrt{x-1}}{x}\,dx = \int_0^1 \frac{t}{t^2+1}(2t\,dt) = 2\int_0^1 \frac{t^2}{t^2+1}\,dt$$

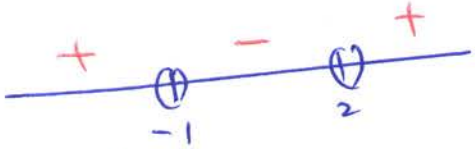
$$\int_1^2 \frac{\sqrt{x-1}}{x}\,dx = 2\int_0^1 \frac{t^2+1-1}{t^2+1}\,dt = 2\int_0^1 \left(1-\frac{1}{t^2+1}\right)dt$$

$$= 2\Big[t - \operatorname{arctg} t\Big]_0^1 = 2\left(1-\operatorname{arctg} 1\right)$$

$$= 2\left(1-\frac{\pi}{4}\right) = 2-\frac{\pi}{2}.$$

(2) $\displaystyle\int_{-3}^{0} |x^2-x-2|\,dx = ?$

Remarquons que le polynôme $x^2-x-2$ possède deux solutions $-1$ et $2$.



Ainsi,

$$\int_{-3}^{0} |x^2-x-2|\,dx = \int_{-3}^{-1} |x^2-x-2|\,dx + \int_{-1}^{0} |x^2-x-2|\,dx$$

$$= \int_{-3}^{-1} (x^2-x-2)\,dx + \int_{-1}^{0} -(x^2-x-2)\,dx$$

$$= \int_{-3}^{-1} (x^2-x-2)\,dx + \int_{-1}^{0} (-x^2+x+2)\,dx$$

$$= \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-3}^{-1} + \left[\frac{-x^3}{3}+\frac{x^2}{2}+2x\right]_{-1}^{0} = \frac{59}{6}$$

(3) $\displaystyle\int_0^1 x^2 \operatorname{arctg} x\,dx = ?$

par une intégration par parties :

$u(x) = \operatorname{arctg} x \quad \Rightarrow \quad u'(x) = \dfrac{1}{1+x^2}$

$v'(x) = x^2 \quad \Rightarrow \quad v(x) = \dfrac{x^3}{3}$

Ainsi, $\displaystyle\int_0^1 x^2 \operatorname{arctg} x\,dx = \left[\frac{x^3}{3}\operatorname{arctg} x\right]_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{x^2+1}\,dx$

$\displaystyle = \frac{1}{3}\operatorname{arctg} 1 - \frac{1}{3}\int_0^1 \left(x - \frac{x}{1+x^2}\right)dx$

$\displaystyle = \frac{\pi}{12} - \frac{1}{3}\left[\frac{x^2}{2} - \frac{1}{2}\ln(1+x^2)\right]_0^1$

$\displaystyle = \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\ln 2.$

(4) $\displaystyle\int_{-1}^1 \frac{dx}{x^2+4x+7} = ?$

Remarquons que $x^2+4x+7 = (x+2)^2 + 3 = 3\left[\left(\frac{x+2}{\sqrt{3}}\right)^2 + 1\right]$

Ainsi, si on pose $t = \frac{x+2}{\sqrt{3}} \Rightarrow dt = \frac{dx}{\sqrt{3}}$ et on a :

$$\int_{-1}^1 \frac{dx}{x^2+4x+7} = \frac{1}{3}\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{\sqrt{3}\,dt}{t^2+1} = \frac{\sqrt{3}}{3}\left[\operatorname{arctg} t\right]_{\frac{\sqrt{3}}{3}}^{\sqrt{3}}$$

$$\left(\begin{array}{l} \text{Si } x=-1, \text{ alors } t = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\ \text{Si } x = 1, \text{ alors } t = \frac{3}{\sqrt{3}} = \sqrt{3} \end{array}\right).$$

Alors

$$\int_{-1}^1 \frac{dx}{x^2+4x+7} = \frac{\sqrt{3}}{3}\left[\operatorname{arctg}\sqrt{3} - \operatorname{arctg}\frac{\sqrt{3}}{3}\right] = \frac{\sqrt{3}}{3}\left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$$= \frac{\pi\sqrt{3}}{18}.$$

Ex 10 :

I) Calculons. $I = \int_0^R \sqrt{R^2 - x^2}\, dx$.

Si $R = 0$, alors $I = 0$,

Si $R < 0$, alors on pose $R' = -R > 0$ et on a :

$$I = \int_0^{-R'} \sqrt{R'^2 - x^2}\, dx.$$

Ainsi, il suffit de calculer $I = \int_0^R \sqrt{R^2 - x^2}\, dx$ où $R > 0$.

posons $x = R \sin t \Rightarrow dx = R \cos t\, dt$.

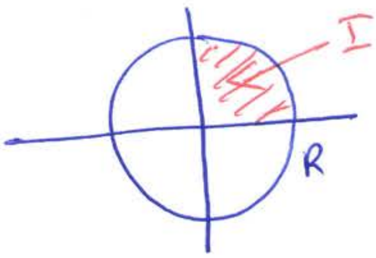
Alors pour $x = 0$, on a : $t = 0$

pour $x = R$ on a $t = \pi/2$.

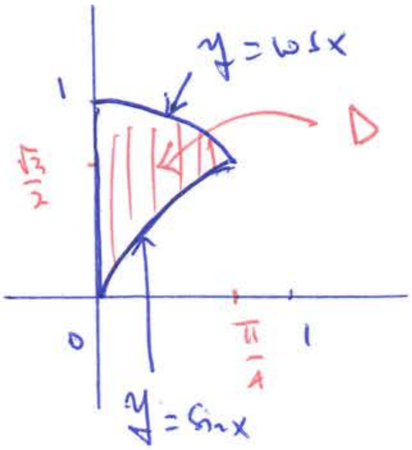
Donc

$$\begin{aligned}
I &= \int_0^{\pi/2} \left(\sqrt{R^2 - R^2 \sin^2 t}\right) (R \cos t)\, dt \\
&= R^2 \int_0^{\pi/2} \left(\sqrt{1 - \sin^2 t}\right)(\cos t\, dt) = R^2 \int_0^{\pi/2} \cos^2 t\, dt \\
&= R^2 \int_0^{\pi/2} \left(\frac{1 + \cos(2t)}{2}\right) dt = \frac{R^2}{2} \int_0^{\pi/2} \left(1 + \cos(2t)\right) dt \\
&= \frac{R^2}{2} \left[ t + \frac{\sin(2t)}{2} \right]_0^{\pi/2} = \frac{\pi R^2}{4}.
\end{aligned}$$

Remarquons que l'intégrale $I$ est l'aire du quart de disque de rayon $R$ et donc l'aire du disque est $4I = 4 \frac{\pi R^2}{4} = \pi R^2$



II) Aire de la région délimitée par les courbes d'équations $y = \cos x$ et $y = \sin x$ entre $0$ et $\frac{\pi}{4}$.




$$\text{Aire}(D) = \int_0^{\pi/4} (\cos x - \sin x)\, dx = \Big[ \sin x + \cos x \Big]_0^{\pi/4}$$
$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1) = \sqrt{2} - 1.$$

Exo 11 : (Facultatif) :

pour $n \in \mathbb{N}$, on a : $I_n = \displaystyle\int_0^{\pi/2} \sin^n x \, dx$.

1) $I_0 = \displaystyle\int_0^{\pi/2} dx = \frac{\pi}{2}$

$I_1 = \displaystyle\int_0^{\pi/2} \sin x \, dx = \Big[-\cos x\Big]_0^{\pi/2} = 1$

et $I_2 = \displaystyle\int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} \left(\frac{1-\cos(2x)}{2}\right) dx = \frac{1}{2}\int_0^{\pi/2} (1-\cos(2x))\, dx$

$= \frac{1}{2}\left[ x - \frac{\sin(2x)}{2}\right]_0^{\pi/2} = \frac{\pi}{4}.$

2) $\forall n \in \mathbb{N}$, $I_{n+1} - I_n = \displaystyle\int_0^{\pi/2} (\sin^{n+1} x - \sin^n x)\, dx$

$= \displaystyle\int_0^{\pi/2} \sin^n x \,(\sin x - 1)\, dx.$

Sachant que

$\forall x \in [0, \frac{\pi}{2}]$ , $0 \leq \sin x \leq 1$ , alors $\sin^n x(\sin x - 1) \leq 0$.

Ainsi, $I_{n+1} - I_n = \int_0^{\pi/2} \sin^n x(\sin x - 1)\,dx \leq 0$

$\Rightarrow (I_n)$ est décroissante.

D'autre part, puisque $(I_n)$ est minorée par 0 et décroissante, alors $(I_n)$ converge.

3) $\forall n \in \mathbb{N}$, $I_{n+2} = \int_0^{\pi/2} \sin^{n+2} x\,dx = \int_0^{\pi/2} \sin x \sin^{n+1} x\,dx$.

On pose

$u(x) = \sin^{n+1} x \Rightarrow u'(x) = (n+1)\cos x \sin^n x$

$v'(x) = \sin x \Rightarrow v(x) = -\cos x$.

Ainsi,

$$I_{n+2} = \underbrace{\left[-\cos x \sin^{n+1} x\right]_0^{\pi/2}}_{0} + \int_0^{\pi/2} (n+1)\cos^2 x \sin^n x\,dx$$

$$= (n+1)\int_0^{\pi/2} (1-\sin^2 x)\sin^n x\,dx$$

$$= (n+1)\int_0^{\pi/2} \sin^n x\,dx - (n+1)\int_0^{\pi/2} \sin^{n+2} x\,dx$$

$$= (n+1)\, I_n - (n+1)\, I_{n+2}.$$

D'où, $I_{n+2} + (n+1) I_{n+2} = (n+1) I_n \Rightarrow (n+2) I_{n+2} = (n+1) I_n$ --- (*)

A) Remarquons que pour

$n=0$, on a : $2I_2 = I_0 \Rightarrow I_2 = \frac{I_0}{2} = \frac{\pi}{2}\cdot\frac{1}{2}$

$n=2$, $4I_4 = 3I_2 \Rightarrow I_4 = \frac{3}{4}I_2 = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$

$$= \frac{4.3.2.1}{4.2.2.4.2}\cdot\frac{\pi}{2} = \frac{(4)!}{2^4(2!)^2} = \frac{(2.2)!}{2^4(2!)^2}$$

$n=4$, $6I_6 = 5I_4 \Rightarrow I_6 = \frac{5}{6}I_4 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$

$$= \frac{(5.3.1)(6.4.2)}{(6.4.2)(6.4.2)}\cdot\frac{\pi}{2}$$

$$= \frac{(6)!}{2^6(3!)^2}\cdot\frac{\pi}{2} = \frac{(2.3)!}{2^{2.3}(3!)^2}$$

$$\ldots\ldots I_{2p} = \frac{\pi}{2}\frac{(2p)!}{(p!)^2\, 2^{2p}} \quad \ldots\ldots (Q_p)$$

Supposons que $(Q_p)$ est vraie pour un $p$ fixé et montrons que $(Q_{p+1})$ est vraie. Alors, d'après la formule $(*)$,

On a :

$$I_{2(p+1)} = I_{2p+2} = \frac{(2p+1)}{2p+2}I_{2p} = \frac{(2p+1)}{2p+2}\frac{\pi}{2}\cdot\frac{(2p)!}{(p!)^2\, 2^{2p}}$$

$$= \frac{(2p+2)(2p+1)(2p)!}{(2p+2)(2p+2)(p!)^2\, 2^{4p}}\quad\frac{\pi}{2}$$

$$= \frac{(2p+2)!}{2^2(p+1)^2(p!)^2\, 2^{2p}}\frac{\pi}{2} = \frac{(2p+2)!}{((p+1)!)^2\, 2^{2p+2}}\cdot\frac{\pi}{2}.$$

Donc, $(Q_p)$ est vérifiée.

De la même manière, on vérifie que $I_{2p+1} = \dfrac{2^{2p}(p!)^2}{(2p+1)!}$

Ex12 : (1) $\displaystyle\int \frac{dx}{x \ln x^2} = \frac{1}{2}\int \frac{dx}{x \ln|x|} = \frac{1}{2}\int \frac{\frac{1}{x}\,dx}{\ln|x|}$.

on pose $u = \ln|x|$, alors on obtient,

$$\frac{1}{2}\int \frac{u'(x)}{u(x)}\,dx = \frac{1}{2}\ln|u(x)| + c, \quad c \in \mathbb{R}$$

$$= \frac{1}{2}\ln\left|\ln|x|\right| + c.$$

(2) $\displaystyle\int \frac{\ln(1+x^2)}{x^2}\,dx = ?$

$u(x) = \ln(1+x^2) \Rightarrow u'(x) = \dfrac{2x}{1+x^2}$

$v'(x) = \dfrac{1}{x^2} \Rightarrow v(x) = -\dfrac{1}{x}$.

On a :

$$\int \frac{\ln(1+x^2)}{x^2}\,dx = -\frac{\ln(1+x^2)}{x} + \int \frac{2\,dx}{1+x^2}$$

$$= -\frac{\ln(1+x^2)}{x} + 2\arctg x + c, \quad c \in \mathbb{R}.$$

(3) $\displaystyle\int \frac{dx}{2\sin^2 x + 3\cos^2 x} = \int \frac{\frac{dx}{\cos^2 x}}{2\frac{\sin^2 x}{\cos^2 x} + 3} = \int \frac{\frac{1}{\cos^2 x}}{2tg^2 x + 3}\,dx$

posons $u = tg\,x \Rightarrow du = \dfrac{dx}{\cos^2 x}$.

Ainsi,

$$\int \frac{dx}{2\sin^2 x+3\cos^2 x} = \int \frac{du}{2u^2+3} = \frac{1}{3}\int \frac{du}{\frac{2}{3}u^2+1}$$

$$= \frac{1}{3}\int \frac{du}{\left(\frac{\sqrt{2}}{\sqrt{3}}u\right)^2+1}$$

On pose

$t = \sqrt{\frac{2}{3}}\, u \Rightarrow dt = \sqrt{\frac{2}{3}}\, du$. Alors:

$$\int \frac{dx}{2\sin^2 x+3\cos^2 x} = \frac{1}{3}\sqrt{\frac{3}{2}}\int \frac{dt}{1+t^2} = \frac{1}{3}\sqrt{\frac{3}{2}}\, \operatorname{arctg} t + c, \quad c \in \mathbb{R}$$

$$= \frac{1}{3}\sqrt{\frac{3}{2}}\, \operatorname{arctg}\left(\sqrt{\frac{2}{3}}\, \operatorname{tg} x\right) + c.$$

(4) $\displaystyle\int_e^{e^2} \frac{dx}{x \ln^n x}$, $n \in \mathbb{N}^*$.

posons $t = \ln x$, d'où $dt = \frac{dx}{x}$.

pour $x = e \Rightarrow t = 1$

pour $x = e^2 \Rightarrow t = \ln e^2 = 2\ln e = 2$.

Ainsi,

$$\int_e^{e^2} \frac{dx}{x \ln^n x} = \int_1^2 \frac{dt}{t^n}.$$

Si $n=1$, alors $\displaystyle\int_1^2 \frac{dt}{t} = \left[\ln t\right]_1^2 = \ln 2$.

Si $n > 1$, alors

$$\int_1^2 \frac{dt}{t^n} = \int_1^2 t^{-n}\, dt = \left[\frac{t^{-n+1}}{-n+1}\right]_1^2 = \frac{1}{1-n}\left[2^{-n+1} - 1\right]$$

⑤ $\displaystyle\int_0^{\pi/4} \frac{\sin(2x)}{1+\cos x}\,dx = \int_0^{\pi/4} \frac{2\sin x\cos x}{1+\cos x}\,dx$

posons $u = \cos x \Rightarrow du = -\sin x\,dx$

pour $x = 0$, $u = 1$

pour $x = \frac{\pi}{4}$, $u = \frac{\sqrt{2}}{2}$.

Ainsi,

$$\int_0^{\pi/4} \frac{2\sin x\cos x}{1+\cos x}\,dx = \int_1^{\sqrt{2}/2} \frac{-2u\,du}{1+u} = 2\int_{\sqrt{2}/2}^{1} \frac{u}{1+u}\,du$$

$$= 2\int_{\sqrt{2}/2}^{1} \left(1 - \frac{1}{1+u}\right) du = 2\left[u - \ln(1+u)\right]_{\frac{\sqrt{2}}{2}}^{1}$$

$$= 2\left[(1-\ln 2) - \left(\frac{\sqrt{2}}{2} - \ln\left(1+\frac{\sqrt{2}}{2}\right)\right)\right]$$

$$= 2 - \sqrt{2} - 4\ln 2 + 2\ln(\sqrt{2}+2).$$