Université Aboubekr Belkaid-Tlemcen

Module: Mathématiques 2
1ère Année Sciences et technologies

Série de TD 02 " Les intégrales"

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Exercice 01: Calculons les primitives suivantes:

$$I_1 = \int \cos 7x dx = \frac{1}{7} \sin 7x + c, c \in \mathbb{R}.$$

$$I_2 = \int \sin x \left(\cos x\right)^3 dx.$$

On pose: $t = \cos x \Rightarrow dt = -\sin x dx$, ce qui donne:

$$I_2 = -\int t^3 dt = -\frac{t^4}{4} + c = -\frac{\left(\cos x\right)^4}{4} + c, c \in \mathbb{R}.$$

$$I_3 = \int x\sqrt{1-x^2} dx = \left(-\frac{1}{2}\right) \int \left(-2x\right) \left(1-x^2\right)^{\frac{1}{2}} dx.$$

On pose: $t = 1 - x^2 \Rightarrow dt = -2x dx$, ce qui donne:

$$\begin{aligned} I_3 &= \left(-\frac{1}{2}\right) \int t^{\frac{1}{2}} dt = \left(-\frac{1}{2}\right) \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c \\ &= \left(-\frac{1}{2}\right) \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = \left(-\frac{1}{3}\right) \left(1-x^2\right)^{\frac{3}{2}}, c \in \mathbb{R}. \end{aligned}$$

$$I_4 = \int \frac{\ln x}{x} dx.$$

On pose: $t = \ln x \Rightarrow dt = \frac{1}{x} dx$, ce qui donne:

$$I_4 = \int t dt = \frac{t^2}{2} + c = -\frac{\left(\ln x\right)^2}{2} + c, c \in \mathbb{R}.$$

$$I_5 = \int e^x \left(1 + e^x\right)^4 dx.$$

On pose: $t = 1 + e^x \Rightarrow dt = e^x dx$, ce qui donne:

$$I_5 = \int t^4 dt = \frac{t^5}{5} + c = -\frac{\left(e^x\right)^5}{5} + c = \frac{e^{5x}}{5} + c, c \in \mathbb{R}.$$

$$I_6 = \int \frac{dx}{x\left(1+\ln x\right)^2} = \int \frac{\left(1+\ln x\right)^{-2}}{x} dx.$$

On pose: $t = 1 + \ln x \Rightarrow dt = \frac{1}{x} dx$, ce qui donne:

$$I_6 = \int t^{-2} dt = \frac{t^{-2+1}}{-2+1} + c = -\frac{\left(1+\ln x\right)^{-1}}{1} + c = -\frac{1}{1+\ln x} + c, c \in \mathbb{R}.$$

$$I_7 = \int \frac{\arctan x}{1+x^2} dx.$$

On pose: $t = \arctan x \Rightarrow dt = \frac{1}{1+x^2}dx$, ce qui donne:

$$I_7 = \int t dt = \frac{t^2}{2} + c = \frac{(\arctan x)^2}{2} + c, c \in \mathbb{R}.$$

$$\begin{aligned}
I_8 &= \int \frac{(1-\sqrt{x})^2}{\sqrt[3]{x}} dx = \int \left(1 + x - 2\sqrt{x}\right) x^{-\frac{1}{3}} dx \\
&= \int \left(x^{-\frac{1}{3}} + x^{\frac{2}{3}} - 2x^{\frac{1}{2}-\frac{1}{3}}\right) dx = \int \left(x^{-\frac{1}{3}} + x^{\frac{2}{3}} - 2x^{\frac{1}{6}}\right) dx \\
&= \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} - 2.\frac{x^{\frac{1}{6}+1}}{\frac{1}{6}+1} + c \\
&= \frac{3}{2}x^{\frac{2}{3}} + \frac{3}{5}x^{\frac{5}{3}} - \frac{12}{7}x^{\frac{7}{6}} + c, c \in \mathbb{R}.
\end{aligned}$$

Exercice 02: Calculons les primitives suivantes, en utilisant l'intégration par parties:

$$I_1 = \int x^2 \cos x dx.$$

Par parties on pose: $u = x^2 \Rightarrow u' = 2xdx$ et $v' = \cos x \Rightarrow v = \sin x$,

$$I_1 = u(x)v(x) - \int u'(x)v(x)dx$$

$$I_1 = x^2 \sin x - 2\int x \sin x dx,$$

on pose:

$$J_1 = \int x \sin x dx.$$

Par parties on pose: $u = x \Rightarrow u' = dx$ et $v' = \sin x \Rightarrow v = -\cos x$,

$$J_1 = u(x)v(x) - \int u'(x)v(x)dx$$

$$J_1 = -x\cos x + \int \cos x dx = -x\cos x + \sin x,$$

D'où:

$$\begin{aligned}
I_1 &= x^2 \sin x - 2(-x\cos x + \sin x) + c \\
&= \left(x^2 - 2\right)\sin x + 2x\cos x + c, c \in \mathbb{R}.
\end{aligned}$$

$$I_2 = \int x^n \ln x dx.$$

Par parties on pose: $u = \ln x \Rightarrow u' = \frac{1}{x}dx$ et $v' = x^n \Rightarrow v = \frac{x^{n+1}}{n+1}$,

$$I_2 = u(x)v(x) - \int u'(x)v(x)dx$$

$$\begin{aligned}
I_2 &= \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^{n+1}}{n+1}.\frac{1}{x}dx \\
&= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n dx \\
&= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\left(\frac{x^{n+1}}{n+1}\right) + c \\
&= \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + c, c \in \mathbb{R}.
\end{aligned}$$

$$I_3 = \int \arccos x dx.$$

Par parties on pose: $u = \arccos x \Rightarrow u' = -\frac{1}{\sqrt{1-x^2}}dx$ et $v' = 1 \Rightarrow v = x$,

$$I_3 = u(x)v(x) - \int u'(x)v(x)\,dx$$

$$\begin{aligned} I_3 &= x\arccos x + \int \frac{x}{\sqrt{1-x^2}}dx \\ &= x\arccos x - \int \frac{-2x}{2\sqrt{1-x^2}}dx \\ &= x\arccos x - \sqrt{1-x^2} + c, c \in \mathbb{R}. \end{aligned}$$

$$I_4 = \int \frac{x}{(\cos x)^2}dx.$$

Par parties on pose: $u = x \Rightarrow u' = dx$ et $v' = \frac{1}{\cos^2 x} \Rightarrow v = \tan x$,

$$I_4 = u(x)v(x) - \int u'(x)v(x)\,dx$$

$$\begin{aligned} I_4 &= x\tan x - \int \tan x dx = x\tan x + \int \frac{-\sin x}{\cos x}dx \\ &= x\tan x + \ln|\cos x| + c, c \in \mathbb{R}. \end{aligned}$$

$$I_5 = \int x\arctan x dx.$$

Par parties on pose: $u = \arctan x \Rightarrow u' = \frac{1}{1+x^2}dx$ et $v' = x \Rightarrow v = \frac{x^2}{2}$,

$$I_5 = u(x)v(x) - \int u'(x)v(x)\,dx$$

$$\begin{aligned} I_5 &= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx \\ &= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}dx \\ &= \frac{x^2}{2}\arctan x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2}dx \\ &= \frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + c, c \in \mathbb{R}. \end{aligned}$$

$$I_6 = \int \frac{x\arcsin x}{\sqrt{1-x^2}}dx.$$

Par parties on pose: $u = \arcsin x \Rightarrow u' = \frac{1}{\sqrt{1-x^2}}dx$ et $v' = \frac{x}{\sqrt{1-x^2}} \Rightarrow v = -\sqrt{1-x^2}$,

$$I_6 = u(x)v(x) - \int u'(x)v(x)\,dx$$

$$\begin{aligned} I_6 &= -\sqrt{1-x^2} - \frac{1}{2}\int \frac{x^2}{1+x^2}dx \\ &= \frac{x^2}{2}\arcsin x + \int \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}dx \\ &= \frac{x^2}{2}\arcsin x + \int dx \\ &= \frac{x^2}{2}\arctan x + x + c, c \in \mathbb{R}. \end{aligned}$$

Exercice 03: En effectuant un changement de variable, calculer:

$$I_1 = \int \sin(\ln x)\, dx$$

On pose: $t = \ln x \Rightarrow dt = \frac{1}{x}dx \Rightarrow dx = xdt = e^t dt$

$$I_1 = \int (\sin t)\, e^t dt$$

Par parties on pose: $u = e^t \Rightarrow u' = e^t dt$ et $v' = \sin t \Rightarrow v = -\cos t$,

$$I_1 = u(x)\, v(x) - \int u'(x)\, v(x)\, dx$$

$$I_1 = -e^t \cos t + \int e^t \cos t dt,$$

on pose:

$$J_1 = \int e^t \cos t dt.$$

Par parties on pose: $u = e^t \Rightarrow u' = e^t dt$ et $v' = \cos t \Rightarrow v = \sin t$,

$$J_1 = u(x)\, v(x) - \int u'(x)\, v(x)\, dx$$

$$J_1 = e^t \sin t - \int (\sin t)\, e^t = e^t \sin t - I_1,$$

D'où:

$$I_1 = -e^t \cos t + J_1 = -e^t \cos t + e^t \sin t - I_1$$

ce qui implique que:

$$\begin{aligned} 2I_1 &= -e^t \cos t + e^t \sin t \\ I_1 &= \frac{1}{2}\left(-e^t \cos t + e^t \sin t\right) + c, c \in \mathbb{R}. \end{aligned}$$

$$I_2 = \int \frac{\sqrt{x-1}}{x} dx.$$

On pose: $t^2 = x - 1 \Rightarrow dx = 2tdt$ et $x = t^2 + 1$

$$\begin{aligned} I_2 &= 2\int \frac{t}{t^2+1}(tdt) = 2\int \frac{t^2+1-1}{t^2+1} dt \\ &= 2\int dt - 2\int \frac{1}{t^2+1} dt \\ &= 2t - 2\arctan t + c, c \in \mathbb{R} \\ &= 2\sqrt{x-1} - 2\arctan\sqrt{x-1} + c, c \in \mathbb{R} \end{aligned}$$

$$I_3 = \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)^4} dx.$$

On pose: $t = \sin x \Rightarrow dt = \cos x dx$, de plus: si $x = 0 \Rightarrow t = 0$ et si $x = \frac{\pi}{2} \Rightarrow t = 1$,

$$\begin{aligned} I_3 &= \int_0^1 \frac{1}{(1+t)^4} dt = \int_0^1 (1+t)^{-4} dt \\ &= \left[\frac{(1+t)^{-4+1}}{-4+1}\right]_0^1 = -\frac{1}{3}\left[2^{-3} - 1^{-3}\right] = -\frac{1}{3}\left[\frac{1}{8} - 1\right] = \frac{7}{24}. \end{aligned}$$

$$I_4 = \int_{-a}^{a} \sqrt{a^2 - x^2} dx = a \int_{-a}^{a} \sqrt{1 - \left(\frac{x}{a}\right)^2} dx.$$

On pose: $t = \frac{x}{a} \Rightarrow dt = \frac{1}{a} dx \Rightarrow dx = adt$, de plus: si $x = -a \Rightarrow t = -1$ et si $x = a \Rightarrow t = 1$,

$$I_4 = a^2 \int_{-1}^{1} \sqrt{1 - t^2} dt,$$

on pose: $t = \sin z \Rightarrow dt = \cos z dz$, de plus: si $t = -1 \Rightarrow z = -\frac{\pi}{2}$ et si $t = 1 \Rightarrow t = \frac{\pi}{2}$, avec $z = \arcsin t$

$$\begin{aligned} I_4 &= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2 z} \cos z dz = a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 z dz \\ &= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos 2z) dz = \frac{a^2}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz + \frac{a^2}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2z dz \\ &= \frac{a^2}{2} [z]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{a^2}{2} \left[\frac{1}{2} \sin 2z\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{a^2}{2} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi a^2}{2}. \end{aligned}$$

Exercice 04: Calculer les primitives des fractions rationnelles suivantes: **(voir le cours des polynômes, paragraphe décomposition en éléments simples)**

$$I_1 = \int \frac{2x - 1}{(x - 1)(x + 2)(x - 2)} dx$$

$$\frac{2x - 1}{(x - 1)(x + 2)(x - 2)} = \frac{a_1}{x - 1} + \frac{a_2}{x + 2} + \frac{a_3}{x - 2},$$

avec:

$$\begin{aligned} a_1 &= \lim_{x \to 1} \frac{2x - 1}{(x + 2)(x - 2)} = -\frac{1}{3}, a_2 = \lim_{x \to -2} \frac{2x - 1}{(x - 1)(x - 2)} = -\frac{5}{12} \\ \text{et } a_3 &= \lim_{x \to 2} \frac{2x - 1}{(x - 1)(x + 2)} = \frac{3}{4}. \end{aligned}$$

D'où:

$$\begin{aligned} I_1 &= \int \left(\frac{a_1}{x - 1} + \frac{a_2}{x + 2} + \frac{a_3}{x - 2}\right) dx \\ &= -\frac{1}{3} \int \frac{1}{x - 1} dx - \frac{5}{12} \int \frac{1}{x + 2} dx + \frac{3}{4} \int \frac{1}{x - 2} dx \\ &= -\frac{1}{3} \ln|x - 1| - \frac{5}{12} \ln|x + 2| + \frac{3}{4} \ln|x - 2| + c, c \in \mathbb{R}. \end{aligned}$$

$$I_2 = \int \frac{x}{(x^4 - 1)} dx.$$

$$\begin{aligned} \frac{x}{(x^4 - 1)} &= \frac{x}{(x^2 - 1)(x^2 + 1)} = \frac{x}{(x - 1)(x + 1)(x^2 + 1)} \\ &= \frac{a_1}{x - 1} + \frac{a_2}{x + 1} + \frac{a_3 x + a_4}{x^2 + 1}, \end{aligned}$$

avec:

$$\begin{aligned} a_1 &= \lim_{x \to 1} \frac{x}{(x + 1)(x^2 + 1)} = \frac{1}{4}, a_2 = \lim_{x \to -1} \frac{x}{(x - 1)(x^2 + 1)} = \frac{1}{4} \\ \text{et } a_3 i + a_4 &= \lim_{x \to i} \frac{x}{(x - 1)(x + 1)} = \frac{i}{(i^2 - 1)} = -\frac{1}{2} i. \end{aligned}$$

(le i est le nombre complexe tel que le dénoménateur s'annule), par identification: $a_3 = -\frac{1}{2}$ et $a_4 = 0$.

$$\begin{aligned} I_2 &= \int \left(\frac{a_1}{x-1} + \frac{a_2}{x+1} + \frac{a_3 x + a_4}{x^2+1} \right) dx \\ &= \frac{1}{4} \int \frac{dx}{x-1} + \frac{1}{4} \int \frac{1}{x+1} dx - \frac{1}{2} \int \frac{1}{x^2+1} dx \\ &= \frac{1}{4} \ln|x-1| + \frac{1}{4} \ln|x+1| - \frac{1}{2} \arctan x + c, c \in \mathbb{R}, \\ &= \frac{1}{4} \ln|(x-1)(x+1)| - \frac{1}{2} \arctan x + c, c \in \mathbb{R}. \end{aligned}$$

$$I_3 = \int \frac{x^3 + 2x^2 + 3x - 1}{(x-1)^2} dx.$$

$$\begin{aligned} \frac{x^3 + 2x^2 + 3x - 1}{(x-1)^2} &= \frac{x^3 + 2x^2 + 3x - 1}{x^2 - 2x + 1} = x + 4 + \frac{10x - 5}{x^2 - 2x + 1} \\ &= x + 4 + \frac{10x - 5}{(x-1)^2}, \end{aligned}$$

par la division euclidienne

$$\begin{array}{|l|l|} x^3 + 2x^2 + 3x - 1 & x^2 - 2x + 1 \\ -x^3 + 2x^2 - x & x + 4 \\ 4x^2 + 2x - 1 & \\ -4x^2 + 8x - 4 & \\ 10x - 5 & \end{array}$$

$$\begin{aligned} I_3 &= \int x + 4 + \frac{10x - 5}{(x-1)^2} dx \\ &= \int x dx + 4 \int dx + \int \frac{10x - 10 + 10 - 5}{(x-1)^2} dx \\ &= \frac{x^2}{2} + 4x + 10 \int \frac{1}{x-1} dx + 5 \int \frac{dx}{(x-1)^2} \\ &= \frac{x^2}{2} + 4x + 10 \ln|x-1| - \frac{5}{x-1} + c, c \in \mathbb{R}. \end{aligned}$$

$$I_4 = \int \frac{x^2 + x + 1}{x^2 - x + 1} dx.$$

Remarquons que le $\triangle = 1 - 4 = -3 < 0$ du polynôme $x^2 - x + 1 = 0$.

$$\begin{aligned} \frac{x^2 + x + 1}{x^2 - x + 1} &= \frac{x^2 - x + 1 + 2x}{x^2 - x + 1} = 1 + \frac{2x}{x^2 - x + 1} \\ &= 1 + \frac{2x - 1 + 1}{x^2 - x + 1} = 1 + \frac{2x - 1}{x^2 - x + 1} + \frac{1}{x^2 - x + 1} \end{aligned}$$

$$\begin{aligned} I_4 &= \int \left(1 + \frac{2x - 1}{x^2 - x + 1} + \frac{1}{x^2 - x + 1} \right) dx \\ &= \int dx + \int \frac{2x - 1}{x^2 - x + 1} dx + \int \frac{1}{x^2 - x + 1} dx \\ &= x + \ln\left(x^2 - x + 1\right) + \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} dx \end{aligned}$$

car:

$$\frac{1}{x^2 - x + 1} = \frac{1}{x^2 - x + \frac{1}{4} - \frac{1}{4} + 1} = \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}.$$

$$I_4 = x + \ln\left(x^2 - x + 1\right) + \frac{1}{\frac{3}{4}} \int \frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1} dx,$$

pour

$$J_4 = \int \frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1} dx$$

on pose: $t = \frac{2x-1}{\sqrt{3}} \Rightarrow dt = \frac{2}{\sqrt{3}} dx \Rightarrow dx = \frac{\sqrt{3}}{2} dt$

$$J_4 = \frac{\sqrt{3}}{2} \int \frac{1}{t^2+1} dt = \frac{\sqrt{3}}{2} \arctan t = \frac{\sqrt{3}}{2} \arctan\left(\frac{2x-1}{\sqrt{3}}\right)$$

ce qui implique que:

$$\begin{aligned} I_4 &= x + \ln\left(x^2 - x + 1\right) + \frac{4}{3}\frac{\sqrt{3}}{2} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + c, c \in \mathbb{R} \\ &= x + \ln\left(x^2 - x + 1\right) + \frac{2}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + c, c \in \mathbb{R}. \end{aligned}$$

Exercice 05: Calculer:

$$\begin{aligned} I_1 &= \int \cos^3 x dx = \int \cos^2 x \cos x \; dx = \int \left(1 - \sin^2 x\right) \cos x \; dx = \int \cos x \; dx - \int \sin^2 x \left(\cos x\right) dx \\ &= \sin x - \frac{1}{3} \sin^3 x + c, c \in \mathbb{R}. \end{aligned}$$

$$\begin{aligned} I_2 &= \int \cos^2 x \sin^2 x dx = \int \left(\cos x \sin x\right)^2 dx = \int \left(\frac{1}{2} \sin 2x\right)^2 dx \\ &= \frac{1}{4} \int \sin^2 \left(2x\right) dx, \end{aligned}$$

on pose: $t = 2x \Rightarrow dt = 2dx \Rightarrow dx = \frac{1}{2} dt$

$$\begin{aligned} I_2 &= \frac{1}{8} \int \sin^2 t dt = \frac{1}{8} \int \frac{1}{2} \left(1 - \cos 2t\right) dt \\ &= \frac{1}{16} \int dt - \frac{1}{16} \int \cos 2t dt \\ &= \frac{t}{16} - \frac{1}{16} \left(\frac{1}{2} \sin 2t\right) + c, c \in \mathbb{R} \\ &= \frac{x}{8} - \frac{1}{32} \sin 4x + c, c \in \mathbb{R}. \end{aligned}$$

$$I_3 = \int \cos 2x \sin 3x dx.$$

On a:

$$\cos a \sin b = \frac{1}{2} \left[\sin\left(b - a\right) + \sin\left(a + b\right)\right],$$

donc:

$$\begin{aligned} I_3 &= \int \frac{1}{2} \left[\sin\left(3x - 2x\right) + \sin\left(3x + 2x\right)\right] dx \\ &= \frac{1}{2} \int \sin x dx + \frac{1}{2} \int \sin 5x dx \\ &= -\frac{1}{2} \cos x - \frac{1}{10} \cos 5x + c, c \in \mathbb{R}. \end{aligned}$$

$$I_4=\int\frac{2\sin x+3\cos x}{3\sin x+2\cos x}dx$$

1ère méthode: On pose: **(voir chapitre divers changements de variables dans les intégrales)**

$$\begin{aligned}x &= 2\arctan z\Rightarrow dx=\frac{2dz}{1+z^2},\left(z=\tan\frac{x}{2}\right)\\ \text{avec} &: \sin x=\frac{2z}{1+z^2}\text{ et }\cos x=\frac{1-z^2}{1+z^2}.\end{aligned}$$

$$\begin{aligned}I_4 &= \int\frac{2.\frac{2z}{1+z^2}+3\frac{1-z^2}{1+z^2}}{3.\frac{2z}{1+z^2}+2\frac{1-z^2}{1+z^2}}\times\frac{2dz}{1+z^2}=2\int\frac{-3z^2+4z+3}{-2z^2+6z+2}\times\frac{dz}{1+z^2}\\ &= \int\frac{3z^2-4z-3}{z^2-3z-1}\times\frac{dz}{1+z^2}=\int\frac{3z^2-4z-3}{(z-z_1)(z-z_2)(1+z^2)}dz\end{aligned}$$

avec:

$$z_1=\frac{3-\sqrt{13}}{2}\text{ et }z_2=\frac{3+\sqrt{13}}{2}.$$

Donc:

$$\begin{aligned}I_4 &= \int\frac{a_1}{z-z_1}dz+\int\frac{a_2}{z-z_2}dz+\int\frac{a_3z+a_4}{1+z^2}dz\\ &= a_1\ln|z-z_1|+a_2\ln|z-z_2|+\frac{a_3}{2}\int\frac{2z}{1+z^2}dz+a_4\int\frac{1}{1+z^2}dz\\ &= a_1\ln|z-z_1|+a_2\ln|z-z_2|+\frac{a_3}{2}\ln\left(1+z^2\right)+a_4\arctan z+c,c\in\mathbb{R}.\\ &= a_1\ln\left|\tan\left(\frac{x}{2}\right)-z_1\right|+a_2\ln\left|\tan\left(\frac{x}{2}\right)-z_2\right|+\frac{a_3}{2}\ln\left(1+\tan^2\left(\frac{x}{2}\right)\right)+a_4\frac{x}{2}+c,c\in\mathbb{R}.\end{aligned}$$

Notons que:

$$\begin{aligned}a_1 &= \lim_{z\to z_1}\frac{3z^2-4z-3}{(z-z_2)(1+z^2)},a_2=\lim_{z\to z_2}\frac{3z^2-4z-3}{(z-z_1)(1+z^2)},\\ \text{et }a_3i+a_4 &= \lim_{z\to i}\frac{3z^2-4z-3}{(z-z_1)(z-z_2)}=\frac{i}{(i^2-1)}=-\frac{1}{2}i.\end{aligned}$$

2ème méthode:

$$I_4=\int\frac{2\sin x+3\cos x}{3\sin x+2\cos x}dx=\int\frac{\cos x}{\cos x}\times\frac{2\frac{\sin x}{\cos x}+3}{3\frac{\sin x}{\cos x}+2}dx=\int\frac{2\tan x+3}{3\tan x+2}dx.$$

On pose: $t=\tan x\Rightarrow dt=\left(1+\tan^2 x\right)dx\Rightarrow dx=\frac{dt}{1+t^2}$, ce qui implique que:

$$\begin{aligned}I_4 &= \int\frac{2t+3}{3t+2}\times\frac{dt}{1+t^2}\\ &= \int\frac{a_1}{3t+2}dt+\int\frac{a_2t+a_3}{1+t^2}dt\\ &= \frac{a_1}{3}\int\frac{3dt}{3t+2}+\frac{a_2}{2}\int\frac{2t}{1+t^2}dt+a_3\int\frac{1}{1+t^2}dt\\ &= \frac{a_1}{3}\ln|3t+2|+\frac{a_2}{2}\ln\left(1+t^2\right)+a_3\arctan t+c,c\in\mathbb{R}.\end{aligned}$$

avec:

$$a_1=\lim_{t\to-\frac{2}{3}}\frac{2t+3}{1+t^2}\text{ et }a_2i+a_3=\lim_{t\to i}\frac{2t+3}{3t+2}.$$

$$I_5=\int\frac{1}{2-\sin^2 x}dx.$$

On pose:

$$\begin{aligned} x &= 2\arctan z \Rightarrow dx = \frac{2dz}{1+z^2}, \left(z = \tan\frac{x}{2}\right) \\ \text{avec} &: \sin x = \frac{2z}{1+z^2}. \end{aligned}$$

$$\begin{aligned} I_5 &= \int \frac{1}{2-\left(\frac{2z}{1+z^2}\right)^2} \times \frac{2dz}{1+z^2} = 2\int \frac{1+z^2}{2+4z^2+2z^4-4z^2}dz \\ &= \int \frac{1+z^2}{1+z^4}dz = \int \frac{1+z^2}{\left(z^2-\sqrt{2}z+1\right)\left(z^2+\sqrt{2}z+1\right)}dz \\ &= \int \frac{a_1 z + a_2}{z^2-\sqrt{2}z+1}dz + \int \frac{a_3 z + a_4}{z^2+\sqrt{2}z+1}dz \end{aligned}$$

Puisque:

$$\begin{aligned} f(z) &= \frac{1+z^2}{1+z^4} \text{ est une fonction paire alors:} \\ a_1 &= a_3 = 0 \text{ et } a_2 = a_4 = \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} I_5 &= \frac{1}{2}\int \frac{dz}{\left(z-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} + \frac{1}{2}\int \frac{dz}{\left(z+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} \\ &= \int \frac{dz}{\left(\sqrt{2}z-1\right)^2+1} + \int \frac{dz}{\left(\sqrt{2}z+1\right)^2+1} \\ &= \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}z-1\right) + \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}z+1\right) + c, c\in\mathbb{R}. \\ &= \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan\frac{x}{2}-1\right) + \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan\frac{x}{2}+1\right) + c, c\in\mathbb{R}. \end{aligned}$$

$$I_6 = \int \frac{\sin x}{1+\sin x}dx.$$

On pose:

$$\begin{aligned} x &= 2\arctan z \Rightarrow dx = \frac{2dz}{1+z^2}, \left(z = \tan\frac{x}{2}\right) \\ \text{avec} &: \sin x = \frac{2z}{1+z^2}. \end{aligned}$$

$$\begin{aligned} I_6 &= \int \frac{\frac{2z}{1+z^2}}{1+\frac{2z}{1+z^2}}\frac{2dz}{1+z^2} = 4\int \frac{z}{\left(1+z^2\right)\left(z^2+2z+1\right)}dz \\ &= 4\int \frac{z}{\left(1+z^2\right)\left(z+1\right)^2}dz = 4\int \frac{a_1}{z+1}dz + 4\int \frac{a_2}{\left(z+1\right)^2}dz + 4\int \frac{a_3 z + a_4}{z^2+1}dz \\ &= 4a_1 \ln|z+1| - \frac{4a_2}{z+1} + 2a_3 \int \frac{2z}{z^2+1}dz + 4a_4 \int \frac{1}{z^2+1}dz \\ &= 4a_1 \ln|z+1| - \frac{4a_2}{z+1} + 2a_3 \ln\left(z^2+1\right) + 4a_4 \arctan z + c, c\in\mathbb{R} \\ &= 4a_1 \ln\left|\left(\tan\frac{x}{2}\right)+1\right| - \frac{4a_2}{\left(\tan\frac{x}{2}\right)+1} + 2a_3 \ln\left(\left(\tan\frac{x}{2}\right)^2+1\right) + 2a_4 x + c, c\in\mathbb{R} \end{aligned}$$

Notons que:

$$\begin{aligned} a_2 &= \lim_{z\to -1}\frac{z}{(1+z^2)} = -\frac{1}{2} \text{ et } a_3 i + a_4 = \lim_{z\to i}\frac{z}{(z^2+2z+1)} = \frac{1}{2} \\ &\Rightarrow a_3 = 0 \text{ et } a_4 = \frac{1}{2}. \\ \text{Pour } z &= 0 \text{ et par identification on trouve: } 0 = a_1 + a_2 + a_4 \Rightarrow a_1 = 0. \end{aligned}$$

$$I_6 = \frac{2}{\left(\tan\frac{x}{2}\right)+1} + x + c, c \in \mathbb{R}$$

$$I_7 = \int_{-5}^{5} x^2 \arctan x dx$$

La méthode est l'intégration par parties mais dans ce cas:

$$f(x) = x^2 \arctan x \text{ est une fonction impaire, ce qui donne que } I_7 = 0.$$

$$I_8 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos 2x}{\cos^2 x} dx = 2\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{\cos^2 x} dx$$

car: $f(x) = \frac{\cos 2x}{\cos^2 x}$ est une fonction paire.

$$\begin{aligned} I_8 &= 2\int_0^{\frac{\pi}{4}} \frac{\cos^2 x - \sin^2 x}{\cos^2 x} dx \\ &= 2\int_0^{\frac{\pi}{4}} \frac{\cos^2 x - \left(1-\cos^2 x\right)}{\cos^2 x} dx \\ &= 2\int_0^{\frac{\pi}{4}} \frac{2\cos^2 x - 1}{\cos^2 x} dx = 2\int_0^{\frac{\pi}{4}} dx - 2\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} dx \\ &= 2[x]_0^{\frac{\pi}{4}} - 2[\tan x]_0^{\frac{\pi}{4}} = 2\frac{\pi}{4} - 2\left(\tan\frac{\pi}{4} - \tan 0\right) = \frac{\pi}{2} - 2. \end{aligned}$$

Exercice 06:

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x} \mathrm{d}x \text{ et } J = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} \mathrm{d}x$$

1) Calculons, et en déduire que $I = J$, en effet:

$$\begin{aligned} I - J &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x - \sin^2 x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{(\cos x - \sin x)(\sin x + \cos x)}{(\sin x + \cos x)} dx \\ &= \int_0^{\frac{\pi}{2}} (\cos x - \sin x) dx = \sin x + \cos x|_0^{\frac{\pi}{2}} = 1 - 1 = 0 \end{aligned}$$

$\Rightarrow I = J.$

2) Vérifions que: $\forall x \in \mathbb{R}, \cos x + \sin x = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$

En effet: $\forall x \in \mathbb{R}$,

$$\begin{aligned} \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) &= \sqrt{2}\left(\cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4}\right) \\ &= \sqrt{2}\left(\cos x \frac{\sqrt{2}}{2} + \sin x \frac{\sqrt{2}}{2}\right) = \cos x + \sin x. \end{aligned}$$

3) En déduire que:

$$I + J = \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}x}{\cos x}.$$

$$\begin{aligned} I+J &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x+\sin^2 x}{\sin x+\cos x}dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x+\cos x}dx \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)}dx, \text{d'après2)} \end{aligned}$$

$\Rightarrow$

$$I+J=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\cos\left(x-\frac{\pi}{4}\right)}dx,$$

on pose: $y=x-\frac{\pi}{4} \Rightarrow dy=dx \Rightarrow$

$$\begin{aligned} I+J &= \frac{1}{\sqrt{2}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos y}dy, \\ &= \frac{2}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{1}{\cos y}dy, \text{ car la fonction est paire,} \\ &= \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{1}{\cos x}dx \end{aligned}$$

4) Le calcul de $I+J$ et en déduire les valeurs de I et J?

$$I+J=\sqrt{2}\int_0^{\frac{\pi}{4}} \frac{1}{\cos x}dx,$$

on pose: $x=2\arctan z \Rightarrow dx=\frac{2}{1+z^2}dz$ et $\cos x=\frac{1-z^2}{1+z^2}$
et quand: $x=0 \Rightarrow z=0$ et si $x=\frac{\pi}{4} \Rightarrow z=\tan\frac{\pi}{8}$
$\Rightarrow$

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{1}{\cos x}dx &= \int_0^{\tan\frac{\pi}{8}} \frac{1+z^2}{1-z^2}\times\frac{2}{1+z^2}dz \\ &= 2\int_0^{\tan\frac{\pi}{8}} \frac{1}{1-z^2}dz \\ &= \int_0^{\tan\frac{\pi}{8}} \frac{1}{1-z}dz+\int_0^{\tan\frac{\pi}{8}} \frac{1}{1+z}dz \\ &= \left[-\ln(1-z)+\ln(1+z)\right]_0^{\tan\frac{\pi}{8}} = \left[\ln\left(\frac{1+z}{1-z}\right)\right]_0^{\tan\frac{\pi}{8}} \\ &= \ln\left(\frac{1+\tan\frac{\pi}{8}}{1-\tan\frac{\pi}{8}}\right). \end{aligned}$$

Alors:

$$\begin{cases} I-J=0 \\ I+J=\sqrt{2}\ln\left(\frac{1+\tan\frac{\pi}{8}}{1-\tan\frac{\pi}{8}}\right) \end{cases}$$

$\Rightarrow$

$$I=J=\frac{1}{\sqrt{2}}\ln\left(\frac{1+\tan\frac{\pi}{8}}{1-\tan\frac{\pi}{8}}\right).$$

Exercice 07: Calculer l'aire d'un disque de rayon R, à l'aide d'une intégrale définie.

Léquation du disque est: $x^2+y^2=R^2$.Ici $f(x)=\sqrt{R^2-x^2}$, l'équation de la partie supériere du disque.

$$\begin{aligned} S &= 2\int_{-R}^{R} \sqrt{R^2-x^2}dx \\ &= 2R\int_{-R}^{R} \sqrt{1-\left(\frac{x}{R}\right)^2}dx, \end{aligned}$$

par la même méthode de l'intégrale I_4 de l'exercice 03, on trouve:

$$S = 2\frac{\pi R^2}{2} = \pi R^2.$$