Corrigé du partiel d'algèbre 3 du 01/12/2015.

exercice n°5 : voir cours.

Ex1:

1/ $$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ a-b & b & c \\ -(a-b) & c+a & a+b \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ -(a-b) & -(b-c) & a+b \end{vmatrix}$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ -1 & -1 & a+b \end{vmatrix} = -1+1 = 0 \implies D_1 = 0.$$

$$D_2 = \begin{vmatrix} a & c & c & b \\ c & a & b & c \\ c & b & a & c \\ b & c & c & a \end{vmatrix} = \begin{vmatrix} a+b+2c & c & c & b \\ a+b+2c & a & b & c \\ a+b+2c & b & a & c \\ a+b+2c & c & c & a \end{vmatrix} = (a+b+2c)\begin{vmatrix} 1 & c & c & b \\ 1 & a & b & c \\ 1 & b & a & c \\ 1 & c & c & a \end{vmatrix}$$

$$= (a+b+2c)\begin{vmatrix} 1 & c & c & b \\ 0 & a-c & b-c & c-b \\ 0 & b-c & a-c & c-b \\ 0 & 0 & 0 & a-b \end{vmatrix} = (a+b+2c)(a-b)\begin{vmatrix} a-c & b-c \\ b-c & a-c \end{vmatrix}.$$

$$= (a+b+2c)(a-b)\left[(a-c)^2 - (b-c)^2\right] \implies D_2 = (a+b+2c)(a-b)^2(a+b-2c)$$

2/ $$D_n = \begin{vmatrix} a & b & \cdots & b \\ b & a & & \vdots \\ \vdots & b & \ddots & \\ & & & b \\ b & b & & a \end{vmatrix} = \begin{vmatrix} a+(n-1)b & b & \cdots & b \\ a+(n-1)b & a & & b \\ \vdots & & b & \vdots \\ & & \vdots & b \\ a+(n-1)b & b & & a \end{vmatrix} = (a+(n-1)b)\begin{vmatrix} 1 & b & & b \\ 1 & a & & b \\ \vdots & & \ddots & \\ 1 & & & a \end{vmatrix}$$

$$= (a+(n-1)b)\begin{vmatrix} 1 & b & & b \\ 0 & a-b & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & a-b \end{vmatrix} = (a+(n-1)b)(a-b)^{n-1}.$$

$$\implies D_n = (a+(n-1)b)(a-b)^{n-1}.$$

Ex2:

1/ $$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \implies \det A = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1-1 = 0$$

$\delta = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$ le système est compatible ssi $\begin{vmatrix} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \\ 1 & 0 & 0 & d \end{vmatrix} = 0$.

c'est à dire $\begin{vmatrix} 1 & 1 & b \\ 0 & 1 & c \\ 0 & 0 & d \end{vmatrix} - \begin{vmatrix} 1 & 0 & a \\ 1 & 1 & b \\ 0 & 1 & c \end{vmatrix} = 0.$

soit $d - (c-b) + a = 0$

d'où le système est compatible ssi $a + b + d - c = 0$.

2°/ $A = \begin{pmatrix} 1 & m & 1 \\ m & 1 & m-1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow \det A = 1 - m.$

1er cas : si $m \neq 1$, $\det A \neq 0$

$\Rightarrow x = \dfrac{\begin{vmatrix} 1 & m & 1 \\ m & 1 & m-1 \\ 1+m & 1 & 1 \end{vmatrix}}{1-m}$; $y = \dfrac{\begin{vmatrix} 1 & 1 & 1 \\ m & m & m-1 \\ 1 & 1+m & 1 \end{vmatrix}}{1-m}$; $z = \dfrac{\begin{vmatrix} 1 & m & 1 \\ m & 1 & m \\ 1 & 1 & 1+m \end{vmatrix}}{1-m}$

2ème cas si $m = 1$ $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ $\det A = 0$. $\delta = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

et $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix} = -1 \neq 0 \Rightarrow$ le système n'a pas de solutions.

Exercice n°3 : Déterminons $M(f)_{1,x,\dots x^n}$.

$f(1) = 1 \quad f(x) = 1 + x \quad f(x^2) = 0 \times 1 + 2x + 1x^2, \dots \quad f(x^n) = 0 \times 1 + 0 \times x + \dots + nx^{n-1} + x^n$

$$\Rightarrow M(f)_{1,x\dots x^n} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{array} \begin{array}{c} \begin{array}{ccccc} f(1) & f(x) & f(x^2) & \dots & f(x^n) \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & & 0 \\ 0 & 1 & 2 & & \vdots \\ \vdots & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & 0 & \ddots & n \\ 0 & 0 & \vdots & & 1 \end{pmatrix} \end{array} \Rightarrow \det f = \det M(f)_{1,x\dots x^n} = 1$$

Exercice n°4

$\dim(F+G) = \dim F + \dim G - \dim(F \cap G)$.

$\Rightarrow \dim(F \cap G) = \dim F + \dim G - \dim(F+G)$

$\dim F + \dim G > n$ et $\dim(F+G) \leq n$ ($(F+G)$ sev de E)

$\Rightarrow \dim F + \dim G > n$ et $-\dim(F+G) \geq -n$

$\Rightarrow \dim(F \cap G) > n - n \Rightarrow \dim(F \cap G) > 0 \Rightarrow \dim(F \cap G) \neq 0$

$\Rightarrow F \cap G \neq \{0\}$